Math 115 Exam #3 Practice Problems

1. Solve the initial-value problem $\frac{dx}{dt} + 2tx = x$, $x(0) = 5$. Use your solution to compute $x(3)$.

   Answer: Subtracting $2tx$ from both sides and factoring the x on the right hand side yields the separable equation
$$\frac{dx}{dt} = x(1 - 2t).$$
   Hence, we can separate variables and integrate
$$\int \frac{dx}{x} = \int (1 - 2t)dt.$$
   Therefore,
$$\ln|x| = t - t^2 + C.$$
   Exponentiating both sides yields
$$|x| = e^{t-t^2+C} = Ae^{t-t^2}.$$
   By allowing A to be negative, we can eliminate the absolute value signs on the left hand side, so
$$x = Ae^{t-t^2}.$$
   Plugging in $t = 0$ yields
$$5 = Ae^{0-0^2} = A,$$
   so we have that
$$x = 5e^{t-t^2}.$$
   Therefore,
$$x(3) = 5e^{3-3^2} = 5e^{-6} = \frac{5}{e^6}.$$

2. Solve the differential equation $7yy' = 5x$.

   Answer: In this equation the variable are already separated, so we can integrate both sides:
$$\int 7ydy = \int 5xdx.$$
   Thus,
$$\frac{7}{2}y^2 = \frac{5}{2}x^2 + C.$$
   Therefore,
$$y^2 = \frac{2}{7}\left(\frac{5}{2}x^2 + C\right) = \frac{5}{7}x^2 + C',$$
   so
$$y = \pm\sqrt{\frac{5}{7}x^2 + C'}.$$

3. Solve the initial-value problem $y' + y = 2$, $y(0) = 1$.

   Answer: This is a linear equation with $P(x) = 1$ and $Q(x) = 2$. Therefore,
$$\int P(x)dx = \int 1dx = x,$$
   so the integrating factor
$$\mu(x) = e^{\int P(x)dx} = e^x.$$

In turn, this means that

$$\int \mu(x)Q(x)dx = \int 2e^x dx = 2e^x + C.$$

Hence,

$$y = \frac{1}{\mu(x)} \int \mu(x)Q(x)dx = \frac{1}{e^x}\left(2e^x + C\right) = 2 + \frac{C}{e^x}.$$

Plugging in $x = 0$ yields

$$1 = 2 + \frac{C}{e^0} = 2 + C,$$

so $C = -1$. Therefore, the solution is

$$y = 2 - \frac{1}{e^x}.$$

(Note: the given equation is also separable: $y' = 2-y$, so could separate and integrate as $\int \frac{dy}{2-y} = \int dx$).

4. Solve the initial-value problem $\frac{dy}{dx} = 1 - y + x^2 - yx^2$, $y(0) = 0$.

   Answer: We can factor the right hand side as $(1-y)(1+x^2)$, so this equation is equivalent to the separable equation

$$\frac{dy}{dx} = (1-y)(1+x^2).$$

   Separate variables and integrate:

$$\int \frac{dy}{1-y} = \int (1+x^2)dx.$$

   Hence,

$$-\ln|1-y| = x + \frac{x^3}{3} + C.$$

   Multiplying both sides by -1 and exponentiating yields

$$|1-y| = e^{-x-x^3/3-C} = Ae^{-x-x^3/3}.$$

   Allowing A to be positive or negative allows us to eliminate the absolute value signs, so

$$1 - y = Ae^{-x-x^3/3}.$$

   Therefore,

$$y = 1 - Ae^{-x-x^3/3}.$$

   Plugging in $x = 0$ yields

$$0 = 1 - Ae^{-0-0^3/3} = 1 - A,$$

   so $A = 1$. Therefore,

$$y = 1 - e^{-x-x^3/3}.$$

5. Solve the differential equation $x\frac{dy}{dx} = y^2$.

   Answer: This is a separable equation, so we can separate variables and integrate:

$$\int \frac{dy}{y^2} = \int \frac{dx}{x}.$$

   Hence,

$$-\frac{1}{y} = \ln|x| + C.$$

   Solving for y, we see that

$$y = -\frac{1}{\ln|x| + C}.$$

6. Solve the differential equation $(x^2+1)\frac{dy}{dx} = y$.

   Answer: This is a separable equation, so we can separate variables and integrate:

$$\int \frac{dy}{y} = \int \frac{dx}{x^2+1}.$$

   Hence,

$$\ln|y| = \tan^{-1} x + C.$$

   Exponentiating both sides yields,

$$|y| = e^{\tan^{-1} x + C} = Ae^{\tan^{-1} x}.$$

   Allowing A to be either positive or negative allows us to eliminate the absolute value signs, so

$$y = Ae^{\tan^{-1} x}.$$

7. Solve the initial-value problem $\frac{dy}{dx} = xy$, $y(1) = 3$.

   Answer: This is a separable equation, so we can separate variables and integrate:

$$\int \frac{dy}{y} = \int x dx.$$

   Thus,

$$\ln|y| = \frac{x^2}{2} + C.$$

   Exponentiating both sides yields

$$|y| = e^{x^2/2+C} = Ae^{x^2/2}.$$

   Allowing A to be either positive or negative lets us get rid of the absolute value signs, so

$$y = Ae^{x^2/2}.$$

   Plugging in $x = 1$, we see that

$$3 = Ae^{1^2/2} = Ae^{1/2} = A\sqrt{e}.$$

   Therefore, $A = \frac{3}{\sqrt{e}}$, so the solution is

$$y = \frac{3}{\sqrt{e}}e^{x^2/2}.$$

8. In a second-order chemical reaction, the reactant A is used up in such a way that the amount of it present decreases at a rate proportional to the square of the amount present. Suppose this reaction begins with 50 grams of A present, and after 10 seconds there are only 25 grams left. How long after the beginning of the reaction will there be only 10 grams left? Will all of the A disappear in a finite time, or will there always be a little bit present?

   Answer: Let $A(t)$ denote the amount of the reactant present after t seconds. Since the amount present decreases at a rate proportional to the amount present,

$$\frac{dA}{dt} = kA^2.$$

   This is a separable equation, so we separate variables and integrate:

$$\int \frac{dA}{A^2} = \int k dt.$$

Therefore,

$$-\frac{1}{A} = kt + C,$$

so, solving for A,

$$A = -\frac{1}{kt + C}.$$

Plugging in $t = 0$, we have that

$$50 = -\frac{1}{k(0) + C} = -\frac{1}{C},$$

so $C = -\frac{1}{50}$. In turn, plugging in $t = 10$ yields

$$25 = -\frac{1}{k(10) - \frac{1}{50}} = -\frac{1}{10k - \frac{1}{50}}.$$

Hence,

$$10k - \frac{1}{50} = -\frac{1}{25},$$

meaning that

$$10k = -\frac{1}{25} + \frac{1}{50} = -\frac{1}{50},$$

so

$$k = -\frac{1}{500}.$$

Therefore, the amount of reactant A present after t seconds is

$$A(t) = -\frac{1}{-\frac{t}{500} - \frac{1}{50}} = -\frac{1}{\frac{-t-10}{500}} = -\frac{500}{-t - 10} = \frac{500}{t + 10}.$$

The time t_0 after which there will only be 10 grams left is determined implicitly by

$$10 = \frac{500}{t_0 + 10},$$

so there will be 10 grams left after $t_0 = 40$ seconds.

Since $A(t) = \frac{500}{t+10}$ and the numerator can never be zero, the amount of A present is never equal to zero.

9. Market research has shown the price p and weekly sales $S(p)$ of a particular product are related by the following differential equation:

$$\frac{dS}{dp} = -\frac{1}{2}\left(\frac{S}{p+3}\right).$$

If sales amount to 100 units when the price is \$1 (i.e. $S(1) = 100$), what will the weekly sales be if the price is raised to \$6?

Answer: The above is a separable equation, so we can separate variables and integrate:

$$\int \frac{dS}{S} = \int -\frac{1}{2}\frac{dp}{p+3}.$$

Therefore,

$$\ln|S| = -\frac{1}{2}\ln|p+3| + C.$$

Since neither sales nor price can be negative, we don't need the absolute value signs. Hence,

$$\ln S = -\frac{1}{2}\ln(p+3) + C = \ln(p+3)^{-1/2} + C.$$

Exponentiating both sides yields

$$S = e^{\ln(p+3)^{-1/2}+C} = Ae^{\ln(p+3)^{-1/2}} = A(p+3)^{-1/2} = \frac{A}{\sqrt{p+3}}.$$

Plugging in $p = 1$ gives

$$100 = \frac{A}{\sqrt{1+3}} = \frac{A}{\sqrt{4}} = \frac{A}{2},$$

so $A = 200$. Thus,

$$S(6) = \frac{200}{6+3} = \frac{200}{9} = 22\frac{2}{9},$$

so if the price is raised to \$6, 22 units will be sold (since presumably it's not possible to sell $\frac{2}{9}$ of a unit).

10. Solve the initial-value problem $\frac{dy}{dx} = \frac{e^{2x}}{6y^5}$, $y(0) = 1$.

    Answer: This is a separable equation, so separate variables and integrate:

$$\int 6y^5 dy = \int e^{2x} dx.$$

Therefore,

$$y^6 = \frac{1}{2}e^{2x} + C,$$

so

$$y = \pm\sqrt[6]{\frac{1}{2}e^{2x} + C}.$$

Plugging in $x = 0$ gives

$$1 = \pm\sqrt[6]{\frac{1}{2}e^{2(0)} + C} = \pm\sqrt[6]{\frac{1}{2} + C},$$

so $C = \frac{1}{2}$. Therefore,

$$y = \pm\sqrt[6]{\frac{1}{2}e^{2x} + \frac{1}{2}}.$$

11. If $y(x)$ satisfies the differential equation $\frac{dy}{dx} = e^{2x-y}$ and $y(0) = 1$, then what is $y(1/2)$?

    Answer: We can write this equation as

$$\frac{dy}{dx} = e^{2x}e^{-y},$$

which is separable. Separate variables and integrate:

$$\int e^y dy = \int e^{2x} dx.$$

Therefore,

$$e^y = \frac{1}{2}e^{2x} + C,$$

so

$$y = \ln\left(\frac{1}{2}e^{2x} + C\right).$$

Plugging in $x = 0$ gives

$$1 = \ln\left(\frac{1}{2}e^{2(0)} + C\right) = \ln\left(\frac{1}{2} + C\right),$$

so $C = e - 1/2$. Therefore,

$$y(1/2) = \ln\left(\frac{1}{2}e^{2(1/2)} + e - \frac{1}{2}\right) = \ln\left(\frac{1}{2} + e - \frac{1}{2}\right) = \ln e = 1.$$

12. Solve the differential equation $x^2y' - y = 2x^3e^{-1/x}$.

Answer: Divide everything by x^2 to get the linear equation in standard form:

$$y' - \frac{1}{x^2}y = 2xe^{-1/x}.$$

Here $P(x) = -\frac{1}{x^2}$ and $Q(x) = 2xe^{-1/x}$, so

$$\int P(x)dx = \int -\frac{1}{x^2}dx = \frac{1}{x}.$$

Thus,

$$\mu(x) = e^{\int P(x)dx} = e^{1/x}.$$

In turn, this means

$$\int \mu(x)Q(x)dx = \int e^{1/x}2xe^{-1/x}dx = \int 2xdx = x^2 + C.$$

Therefore,

$$y = \frac{1}{\mu(x)}\int \mu(x)Q(x)dx = \frac{1}{e^{1/x}}\left(x^2 + C\right) = \frac{x^2}{e^{1/x}} + \frac{C}{e^{1/x}}.$$

13. Solve the initial-value problem $xy' - y = x\ln x$, $x > 0$, $y(1) = 2$.

Answer: Dividing everything by x yields the linear equation in standard form:

$$y' - \frac{1}{x}y = \ln x.$$

Here $P(x) = -\frac{1}{x}$ and $Q(x) = \ln x$, so

$$\int P(x)dx = \int -\frac{1}{x}dx = -\ln|x| = \ln\left(\frac{1}{x}\right)$$

(where we can eliminate the absolute value signs because $x > 0$). Therefore,

$$\mu(x) = e^{\int P(x)dx} = e^{\ln(1/x)} = \frac{1}{x}.$$

In turn,

$$\int \mu(x)Q(x)dx = \int \frac{1}{x}\ln x dx = \frac{(\ln x)^2}{2} + C,$$

so we have that

$$y = \frac{1}{\mu x}\int \mu(x)Q(x)dx = \frac{1}{\frac{1}{x}}\left(\frac{(\ln x)^2}{2} + C\right) = \frac{x(\ln x)^2}{2} + Cx.$$

Plugging in $x = 1$ yields

$$2 = \frac{1 \cdot (\ln 1)^2}{2} + C \cdot 1 = C,$$

so

$$y = \frac{x(\ln x)^2}{2} + 2x.$$

14. Solve the initial-value problem $(x^2+1)\frac{dy}{dx} + 3x(y-1) = 0$, $y(0) = 2$.

Answer: Dividing everything by x^2+1 yields

$$y' + \frac{3x}{x^2+1}(y-1) = 0.$$

This isn't quite in standard form; to get it into standard form, add $\frac{3x}{x^2+1}$ to both sides:

$$y' + \frac{3x}{x^2+1}y = \frac{3x}{x^2+1}.$$

Here $P(x) = Q(x) = \frac{3x}{x^2+1}$, so

$$\int P(x)dx = \int \frac{3x}{x^2+1}dx = \frac{3}{2}\ln(x^2+1) = \ln(x^2+1)^{3/2},$$

so

$$\mu(x) = e^{\int P(x)dx} = e^{\ln(x^2+1)^{3/2}} = (x^2+1)^{3/2}.$$

Therefore,

$$\int \mu(x)Q(x)dx = \int (x^2+1)^{3/2}\frac{3x}{x^2+1}dx = \int 3x\sqrt{x^2+1}dx = (x^2+1)^{3/2} + C,$$

so the solution is

$$y = \frac{1}{\mu(x)}\int \mu(x)Q(x)dx = \frac{1}{(x^2+1)^{3/2}}\left((x^2+1)^{3/2} + C\right) = 1 + \frac{C}{(x^2+1)^{3/2}}.$$

Plugging in $x = 0$ yields

$$2 = 1 + \frac{C}{(0^2+1)^{3/2}} = 1 + C,$$

so $C = 1$. Therefore,

$$y = 1 + \frac{1}{(x^2+1)^{3/2}}.$$

15. Solve the differential equation $t\ln t\frac{dr}{dt} + r = te^t$ assuming $t > 1$.

Answer: Dividing everything by $t\ln t$ yields the linear equation in standard form

$$r' + \frac{1}{t\ln t}r = \frac{e^t}{\ln t}.$$

Here $P(t) = \frac{1}{t\ln t}$ and $Q(t) = \frac{e^t}{\ln t}$, so

$$\int P(t)dt = \int \frac{1}{t\ln t}dt = \ln|\ln t| = \ln\ln t$$

since $t > 1$. Therefore,

$$\mu(t) = e^{\int P(t)dt} = e^{\ln\ln t} = \ln t$$

and so

$$\int \mu(t)Q(t)dt = \int \ln t\frac{e^t}{\ln t}dt = \int e^t dt = e^t + C.$$

Hence,

$$r = \frac{1}{\mu(t)}\int \mu(t)Q(t)dt = \frac{1}{\ln t}\left(e^t + C\right) = \frac{e^t}{\ln t} + \frac{C}{\ln t}.$$

16. In the following predator-prey system, determine which of the variables, x or y, represents the prey population and which represents the predator population. Do the predators feed only on the prey or do they have additional food sources? Explain.

$$\frac{dx}{dt} = -0.05x + 0.0001xy$$
$$\frac{dy}{dt} = 0.1y - 0.005xy$$

Answer: Since the y population will grow exponentially without x, whereas x will die off without y, y must be the prey population and x must be the predator population.

If the predators had another food source then they wouldn't go extinct in the absence of the prey y; however, it's clear from the given system of equations that the predators *will* die out in the absence of y, so they must not have another food source.