Math 115 Exam #2

1. Find the limit

$$\lim_{x\to 0} \frac{\cos x - 1}{e^{x^2} - 1}.$$

Answer: The Maclaurin series for $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots,$$

so the numerator can be written as

$$\cos x - 1 = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) - 1 = -\frac{x^2}{2!} + \frac{x^4}{4!} - \dots.$$

On the other hand, using the Maclaurin series $1 + x + x^2/2! + x^3/3!$, we can write the Maclaurin series for e^{x^2}:

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots.$$

Therefore, the denominator has Maclaurin series

$$e^{x^2} - 1 = \left(1 + x^2 + \frac{x^4}{2!} + \dots\right) - 1 = x^2 + \frac{x^4}{2!} + \dots.$$

Therefore,

$$\lim_{x\to 0} \frac{\cos x - 1}{e^{x^2} - 1} = \lim_{x\to 0} \frac{-\frac{x^2}{2!} + \frac{x^4}{4!} - \dots}{x^2 + \frac{x^4}{2!} + \dots}.$$

Dividing numerator and denominator by x^2 gives the value of the limit:

$$\lim_{x\to 0} \frac{-\frac{1}{2} + \frac{x^2}{24} - \dots}{1 + \frac{x^2}{2} + \dots} = -\frac{1}{2}.$$

2. Determine the third degree Taylor polynomial centered at $a = 0$ for the function

$$\int_0^x \cos\sqrt{t}\, dt.$$

Show that this Taylor polynomial will estimate the actual value of the function within 0.001 for any x between 0 and 1.

Answer: First, note that the Maclaurin series for $\cos\sqrt{t}$ is

$$\cos\sqrt{t} = 1 - \frac{\sqrt{t}^2}{2!} + \frac{\sqrt{t}^4}{4!} - \frac{\sqrt{t}^6}{6!} + \dots = 1 - \frac{t}{2!} + \frac{t^2}{4!} - \frac{t^3}{6!} + \dots = \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{(2k)!}.$$

Therefore

$$\begin{aligned}
\int_0^x \cos\sqrt{t} dt &= \int_0^x \left(1 - \frac{t}{2!} + \frac{t^2}{4!} - \frac{t^3}{6!} + \dots\right) dt \\
&= \left[t - \frac{t^2}{2\cdot 2!} + \frac{t^3}{3\cdot 4!} - \frac{t^4}{4\cdot 6!} + \dots\right]_0^x \\
&= x - \frac{x^2}{4} + \frac{x^3}{72} - \frac{x^4}{2880} + \dots
\end{aligned}$$

Hence, the third degree Taylor polynomial is

$$x - \frac{x^2}{4} + \frac{x^3}{72}.$$

Since the Taylor series is alternating, the error is no bigger than the next term in the series:

$$|\text{error}| \leq \frac{|x|^4}{2880} \leq \frac{1}{2880} < \frac{1}{1000} = 0.001$$

for any x between 0 and 1.

3. Estimate $\sqrt{4.1}$ to the nearest 0.001 (use the series for $\sqrt{x}$ around $a = 4$).

   Answer: Let $f(x) = \sqrt{x}$. Then the first few derivatives of f are

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x}} \\ f''(x) &= -\frac{1}{4x^{3/2}} \\ f'''(x) &= \frac{3}{8x^{5/2}} \\ f^{(4)}(x) &= -\frac{15}{16x^{7/2}}. \end{aligned}$$

   Hence,

$$\begin{aligned} f(4) &= 2 \\ f'(4) &= \frac{1}{4} \\ f''(4) &= -\frac{1}{32} \\ f'''(4) &= \frac{3}{256} \\ f^{(4)}(x) &= -\frac{15}{2048} \end{aligned}$$

   Therefore, the Taylor series for f centered at 4 is

$$2 + \frac{(x-4)}{4} - \frac{(x-4)^2}{32 \cdot 2!} + \frac{3(x-4)^3}{256 \cdot 3!} - \frac{15(x-4)^4}{2048 \cdot 4!} + \ldots.$$

   When $x = 4.1$, this gives

$$2 + \frac{0.1}{4} - \frac{(0.1)^2}{64} + \frac{3(0.1)^3}{256 \cdot 6} - \frac{15(0.1)^4}{2048 \cdot 24} + \ldots = 2 + \frac{1}{40} - \frac{1}{6400} + \frac{3}{256 \cdot 6000} - \frac{15}{2048 \cdot 240,000} + \ldots$$

   Since this series is alternating past the first term, the error for any finite part of it is less than the next term in the series. Since the third term is $\frac{1}{6400} < \frac{1}{1000} = 0.001$, we can estimate $\sqrt{4.1}$ to within 0.001 by taking just the first two terms in the series:

$$\sqrt{4.1} \approx 2 + \frac{1}{40} = 2.025.$$

4. If $f(x) = x^3 \cos x^2$, then what is $f^{(13)}(0)$?

   [*Hint*: Consider an appropriate series]

Answer: The Maclaurin series for $\cos x^2$ is

$$\cos x^2 = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \ldots = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \ldots.$$

Therefore, the Maclaurin series for f is

$$f(x) = x^3\left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \ldots\right) = x^3 - \frac{x^7}{2!} + \frac{x^{11}}{4!} - \frac{x^{15}}{6!} + \ldots \quad (1)$$

However, by definition of the Maclaurin series,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

So $\frac{f^{(13)}(0)}{13!}$ must be the coefficient on x^{13}. However, in the expression (1), the coefficient on x^{13} is zero. Therefore, it must be the case that

$$f^{(13)}(0) = 0.$$

5. Find a series representation for $\int \frac{e^x}{x} dx$.

   Answer: The Maclaurin series for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

   Therefore, $\frac{e^x}{x}$ can be represented by the series

$$\frac{1}{x} + 1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots.$$

   Hence, the integral is

$$\int \frac{e^x}{x} dx = \int \left(\frac{1}{x} + 1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots\right) dx = C + \ln x + x + \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} + \ldots = C + \ln x + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$$

6. Find all the third roots of -8. Pick one that is not a real number and write it in the form $a + bi$.

   Answer: Suppose α is a third root of -8. If $\alpha = re^{i\theta}$, then

$$-8 = \alpha^3 = \left(re^{i\theta}\right)^3 = r^3 e^{i(3\theta)}$$

   Since -8 can be represented in polar form as

$$8e^{i\pi}, 8e^{i(3\pi)}, 8e^{i(5\pi)},$$

   this means that $r = 2$ and

$$3\theta = \pi, 3\pi, 5\pi.$$

   Hence, the third roots of -8 are

$$2e^{i(\pi/3)}, 2e^{i(3\pi/3)}, 2e^{i(5\pi/3)},$$

   or, simplifying a bit,

$$2e^{i(\pi/3)}, -2, 2e^{i(5\pi/3)}.$$

   For the second part of the problem, consider $2e^{i(\pi/3)}$. Then, by Euler's formula,

$$2e^{i(\pi/3)} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2\frac{1}{2} + 2\frac{\sqrt{3}}{2}i = 1 + \sqrt{3}\,i.$$