Math 115 Exam #2 Practice Problem Solutions

1. Find the Maclaurin series for $\tan^{-1}(x^2)$ (feel free just to write out the first few terms).

   Answer: Let $f(x) = \tan^{-1}(x)$. Then the first few derivatives of f are:

$$\begin{aligned} f'(x) &= \frac{1}{1+x^2} \\ f''(x) &= -\frac{2x}{(1+x^2)^2} \\ f'''(x) &= -\frac{2}{(1+x^2)^2} + \frac{8x^2}{(1+x^2)^3} \\ f^{(4)}(x) &= \frac{8x}{(1+x^2)^3} + \frac{16x}{(1+x^2)^3} - \frac{48x^3}{(1+x^2)^4} \end{aligned}$$

   Therefore,

$$\begin{aligned} f(0) &= \tan^{-1}(0) = 0 \\ f'(0) &= 1 \\ f''(0) &= 0 \\ f'''(0) &= -2 \\ f^{(4)}(0) &= 0 \end{aligned}$$

   so the Maclaurin series for $\tan^{-1}(x)$ starts out as

$$x - \frac{2}{3!}x^3 + \ldots = x - \frac{x^3}{3} + \ldots$$

   Replacing x with x^2, we get that the Maclaurin series for $\tan^{-1}(x^2)$ is

$$x^2 - \frac{x^6}{3} + \ldots$$

2. Use the first two non-zero terms of an appropriate series to give an approximation of

$$\int_0^1 \sin(x^2)dx.$$

   Give (with explanation) an estimate of the error (the difference between your approximation and the actual value of the integral).

   Answer: We know that the Maclaurin series for $\sin x$ is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots,$$

   so the Maclaurin series for $\sin(x^2)$ is

$$x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \ldots.$$

   Therefore,

$$\begin{aligned} \int_0^1 \sin(x^2)dx &= \int_0^1 \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \ldots\right) dx \\ &= \left[\frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \right]_0^1 \\ &= \frac{1}{3} - \frac{1}{42} + \frac{1}{11\cdot 120} - \ldots \end{aligned}$$

Thus, approximating by the first two terms, we see that

$$\int_0^1 \sin(x^2)dx \approx \frac{1}{3} - \frac{1}{42} = \frac{13}{42}.$$

By the Alternating Series Approximation Theorem, the error of this estimate is no bigger than the next term in the series, which is

$$\frac{1}{11 \cdot 120} = \frac{1}{1320},$$

so the error is certainly less than $\frac{1}{1000} = 0.001$.

3. Find the limit

$$\lim_{x \to 0} \frac{\sin x^2}{1 - \cos 2x}$$

without using L'Hôpital's Rule.

Answer: From Problem 2 we know that

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \dots$$

The Maclaurin series for $\cos x$ is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots,$$

so the series for $\cos 2x$ is

$$1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - \dots.$$

Therefore,

$$\lim_{x \to 0} \frac{\sin x^2}{1 - \cos 2x} = \lim_{x \to 0} \frac{x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \dots}{1 - \left(1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - \dots\right)} = \lim_{x \to 0} \frac{x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \dots}{\frac{4x^2}{2!} - \frac{16x^4}{4!} + \dots}.$$

Dividing numerator and denominator by x^2 yields

$$\lim_{x \to 0} \frac{1 - \frac{x^4}{3!} + \frac{x^8}{5!} - \dots}{\frac{4}{2!} - \frac{16x^2}{4!} + \dots} = \frac{1}{\frac{4}{2!}} = \frac{2}{4} = \frac{1}{2}.$$

4. Find the Taylor series for e^{-x^2} centered at 0. What is the interval of convergence for this series?

Answer: The Maclaurin series for e^x is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Therefore, replacing x with $-x^2$, the Maclaurin series for e^{-x^2} is

$$\sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}.$$

To find the interval of convergence, we use the Ratio Test:

$$\lim_{n \to \infty} \left| \frac{(-1)^{n+1} \frac{x^{2n+2}}{(n+1)!}}{(-1)^n \frac{x^{2n}}{n!}} \right| = \lim_{n \to \infty} \frac{|x|^2}{n+1} = |x|^2 \lim_{n \to \infty} \frac{1}{n+1} = 0,$$

which is certainly less than 1, so this series converges absolutely for all x. Therefore, the interval of convergence is

$$(-\infty, \infty).$$

5. Find the Maclaurin series for

$$\int_0^x \cos t^3 dt.$$

Answer: Since the Maclaurin series for $\cos x$ is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

we can replace x with t^3 to get the Maclaurin series for $\cos t^3$:

$$1 - \frac{t^6}{2!} + \frac{t^{12}}{4!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{6n}}{(2n)!}.$$

Therefore,

$$\int_0^x \cos t^3 dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n \frac{t^{6n}}{(2n)!} dt = \left[\sum_{n=0}^{\infty} (-1)^n \frac{t^{6n+1}}{(6n+1)(2n)!} \right]_0^x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(6n+1)(2n)!}$$

where we evaluate the integral term-by-term. The first few terms of this series are

$$x - \frac{x^7}{14} + \frac{x^{13}}{78} - \ldots$$

6. Write out the first five terms of the Taylor series for $\sqrt{x}$ centered at $x = 1$.

Answer: Let $f(x) = \sqrt{x}$ and evaluate the first few derivatives of f:

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x}} \\ f''(x) &= -\frac{1}{4x^{3/2}} \\ f'''(x) &= \frac{3}{8x^{5/2}} \\ f^{(4)}(x) &= -\frac{15}{16x^{7/2}} \end{aligned}$$

Therefore,

$$\begin{aligned} f(1) &= 1 \\ f'(1) &= \frac{1}{2} \\ f''(1) &= -\frac{1}{4} \\ f'''(1) &= \frac{3}{8} \\ f^{(4)}(1) &= -\frac{15}{16} \end{aligned}$$

Hence, the Taylor series for $\sqrt{x}$ centered at 1 is

$$1 + \frac{1}{2}(x-1) - \frac{1}{4 \cdot 2!}(x-1)^2 + \frac{3}{8 \cdot 3!}(x-1)^3 - \frac{15}{16 \cdot 4!}(x-1)^4 + \ldots$$

or, simplifying slightly,

$$1 + \frac{(x-1)}{2} - \frac{(x-1)^2}{8} + \frac{3(x-1)^3}{48} - \frac{15(x-1)^4}{376} + \ldots$$

7. Find the Maclaurin series for $f(x) = \frac{1}{1+2x^2}$. What is its interval of convergence?

   Answer: Writing $\frac{1}{1+2x^2}$ as
$$\frac{1}{1-(-2x^2)},$$
   we can use the geometric series to see that
$$\frac{1}{1+2x^2} = \sum_{n=0}^{\infty}(-2x^2)^n = \sum_{n=0}^{\infty}(-1)^n 2^n x^{2n}.$$
   Since the equality $\frac{1}{1-r} = \sum r^n$ is only valid when $|r| < 1$, we see that this series converges for $|-2x^2| < 1$, meaning that $|x| < 1/\sqrt{2}$ (we could also have seen this by using the Ratio Test), so the interval of convergence is
$$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$
   If it's not clear that the series doesn't converge at the endpoints, it's easy to check the endpoints: when $x = 1/\sqrt{2}$, the series is
$$\sum_{n=0}^{\infty}(-1)^n 2^n \left(\frac{1}{\sqrt{2}}\right)^{2n} = \sum_{n=0}^{\infty}(-1)^n,$$
   which diverges. When $x = -1/\sqrt{2}$, the series is
$$\sum_{n=0}^{\infty}(-1)^n 2^n \left(\frac{-1}{\sqrt{2}}\right)^{2n} = \sum_{n=0}^{\infty} 1,$$
   which also diverges. Therefore, the series diverges at both endpoints and the interval of convergence is as stated above.

8. Plugging in $x = 1$ to the Maclaurin series for e^x, we can write e as
$$e = \sum_{k=0}^{\infty}\frac{1}{k!}.$$
   How many terms are necessary to approximate e to within $\frac{1}{8}$? You may take it as known that $e \leq 3$.

   Answer: Let $f(x) = e^x$. Using the Taylor Approximation Theorem, the partial sum
$$\sum_{k=0}^{n}\frac{1}{k!} = 1 + 1 + \frac{1}{2} + \ldots + \frac{1}{n!}$$
   has an error no larger than
$$\frac{M}{(n+1)!}1^{n+1},$$
   where M is an upper bound on $f^{(n+1)}(x)$ for all x such that $|x| \leq 1$. Since $f^{(n+1)}(x) = e^x$ for any n and since e^x is an increasing function,
$$f^{(n+1)}(x) = e^x \leq e^1 = e$$
   for any x such that $|x| \leq 1$. e is what we're trying to approximate, so it's not a good choice for M, but in the problem we're told we may assume $e < 3$, so 3 is a good choice for M.

   Therefore, the error is no larger than
$$\frac{M}{(n+1)!}1^{n+1} = \frac{3}{(n+1)!}.$$

Hence, if we choose n so that the above quantity is $\leq \frac{1}{8}$, we'll be done. Of course, if $n = 3$, then $(n+1)! = 4! = 24$ and $\frac{3}{24} = \frac{1}{8}$, so we can approximate e to within $\frac{1}{8}$ by

$$\sum_{k=0}^{3} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} = \frac{8}{3}.$$

9. Find all the sixth roots of -1.

Answer: Notice that we can write -1 in polar form as

$$e^{i\pi}.$$

Hence, if $re^{i\theta}$ is a sixth root of -1, then it must be the case that

$$e^{i\pi} = \left(re^{i\theta}\right)^6 = r^6 e^{i(6\theta)},$$

so $r = 1$. Also, we could choose θ such that $6\theta = \pi$, so $\theta = \pi/6$. Of course,

$$e^{i\pi} = e^{i(3\pi)} = e^{i(5\pi)} = e^{i(7\pi)} = e^{i(9\pi)} = e^{i(11\pi)} = \ldots,$$

so we could also choose

$$\theta = \frac{3\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{9\pi}{6}, \frac{11\pi}{6}.$$

(Notice that $e^{i\frac{13\pi}{6}} = e^{i\frac{\pi}{6}}$, so we can stop here).

Since $\frac{3\pi}{6} = \frac{\pi}{2}$ and $\frac{9\pi}{6} = \frac{3\pi}{2}$, the sixth roots of -1 are

$$e^{i\frac{\pi}{6}}, e^{i\frac{\pi}{2}}, e^{i\frac{5\pi}{6}}, e^{i\frac{7\pi}{6}}, e^{i\frac{3\pi}{2}}, e^{i\frac{11\pi}{6}}.$$

10. Write $\sqrt{3} - i$ in polar form.

Answer: We first compute the modulus:

$$|\sqrt{3} - i| = \sqrt{\sqrt{3}^2 + (-1)^2} = \sqrt{3+1} = 2,$$

so $\sqrt{3} - i = 2e^{i\theta}$ for some θ. In particular

$$\theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \frac{5\pi}{6}.$$

Therefore,

$$\sqrt{3} - i = 2e^{i\frac{5\pi}{6}}.$$