Math 115 Exam #1 Solutions

1. Does
$$\sum_{n=0}^{\infty} \frac{2^n + 3^n}{4^n}$$
converge? If so, what is the sum?

   Answer: We can re-write this as the sum of two geometric series:
$$\sum_{n=0}^{\infty} \frac{2^n + 3^n}{4^n} = \sum_{n=0}^{\infty} \frac{2^n}{4^n} + \sum_{n=0}^{\infty} \frac{3^n}{4^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n.$$
   Using what we know about the sums of geometric series, this is equal to
$$\frac{1}{1-\frac{1}{2}} + \frac{1}{1-\frac{3}{4}} = \frac{1}{\frac{1}{2}} + \frac{1}{\frac{1}{4}} = 2 + 4 = 6,$$
   so the sum of the given series is 6.

2. Determine whether the series
$$\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$$
converges or diverges.

   Answer: Do a limit comparison with the series $\sum \frac{1}{n^2}$, which we know converges because it's a p-series with $p = 2 > 1$:
$$\lim_{n\to\infty} \frac{\frac{\sqrt[n]{n}}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}n^2}{n^2} = \lim_{n\to\infty} \sqrt[n]{n} = 1.$$
   Therefore, since $\sum \frac{1}{n^2}$ converges, the Limit Comparison Test says that the given series also converges.

3. Does the series
$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{1+n^2}$$
converge absolutely, converge conditionally, or diverge?

   Answer: The Alternating Series Test will say that the series converges provided we can show that (i) $\lim_{n\to\infty} \frac{n}{1+n^2} = 0$ and (ii) the sequence of terms $\frac{n}{1+n^2}$ are decreasing. To see (i), notice that we can divide numerator and denominator by n^2 to get
$$\lim_{n\to\infty} \frac{\frac{1}{n^2}\cdot n}{\frac{1}{n^2}(1+n^2)} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n^2}+1} = 0.$$
   To see (ii), let $f(x) = \frac{x}{1+x^2}$. Then
$$f'(x) = \frac{(1+x^2)\cdot 1 - x\cdot 2x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} \leq 0$$
   for $x \geq 1$. Therefore, f is a decreasing function in the relevant range, so the terms $f(n) = \frac{n}{1+n^2}$ are decreasing.

We know that the series converges, but we need to determine whether it converges absolutely or not. In other words, we must determine if

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{n}{1+n^2} \right| = \sum_{n=1}^{\infty} \frac{n}{1+n^2}.$$

converges or not. To see that this series diverges, limit compare with the harmonic series $\sum \frac{1}{n}$, which we know diverges:

$$\lim_{n \to \infty} \frac{\frac{n}{1+n^2}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n^2}{1+n^2} = 1.$$

Hence, the Limit Comparison Test says that the series $\sum \frac{n}{1+n^2}$ diverges.

Therefore, the series $\sum (-1)^n \frac{n}{1+n^2}$ converges but does not converge absolutely, so it converges conditionally.

4. How many terms from the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

are needed to approximate the sum within 0.05?

Answer: We will take the partial sum

$$s_k = \sum_{n=1}^{k} \frac{1}{n^3}$$

and we want the remainder

$$R_k = s - s_k = \sum_{n=k+1}^{\infty} \frac{1}{n^3}$$

to be less than $0.05 = \frac{1}{20}$. Now, the remainder estimate for the integral test says that

$$\int_{k+1}^{\infty} \frac{1}{x^3} dx \leq R_k \leq \int_{k}^{\infty} \frac{1}{x^3} dx,$$

so we know that $R_k < \frac{1}{20}$ whenever the integral on the right is less than $\frac{1}{20}$. In other words, we want

$$\frac{1}{20} > \int_{k}^{\infty} \frac{1}{x^3} dx = \left[-\frac{1}{2x^2} \right]_k^{\infty} = \frac{1}{2k^2}.$$

Now, $\frac{1}{2k^2} < \frac{1}{20}$ whenever $20 < 2k^2$ or, equivalently, when

$$10 < k^2.$$

Therefore, adding the first 4 terms will approximate the sum within 0.05.

5. Does the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2 + \sqrt{n}}$$

converge absolutely, converge conditionally, or diverge?

Answer: Note that $|\cos n| \leq 1$ for all n, so

$$\left| \frac{\cos n}{n^2 + \sqrt{n}} \right| = \frac{|\cos n|}{n^2 + \sqrt{n}} \leq \frac{1}{n^2 + \sqrt{n}} < \frac{1}{n^2}$$

for all n. Therefore,

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2 + \sqrt{n}} \right| < \sum_{n=1}^{\infty} \frac{1}{n^2};$$

since the series on the right converges, we see that the original series converges absolutely.

6. Does the series

$$\sum_{n=1}^{\infty} \frac{n3^n(n+2)!}{4^n n!}$$

converge or diverge?

Answer: Using the Ratio Test,

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)3^{n+1}(n+3)!}{4^{n+1}(n+1)!}}{\frac{n3^n(n+2)!}{4^n n!}} \right| = \lim_{n\to\infty} \left| \frac{3(n+3)}{n} \right| = \frac{3}{4} \lim_{n\to\infty} \frac{n+3}{n} = \frac{3}{4}.$$

Since $\frac{3}{4} < 1$, the Ratio Test says that this series converges absolutely.

7. Determine the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^n}{4^n}$$

Answer: Using the Ratio Test,

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2} x^{n+1}}{4^{n+1}}}{\frac{(-1)^{n+1} x^n}{4^n}} \right| = \lim_{n\to\infty} \frac{|x|}{4}.$$

So the series converges absolutely when $\frac{|x|}{4} < 1$ or, equivalently, when

$$|x| < 4.$$

Thus, the radius of convergence is 4 and, to find the interval of convergence, we just need to check what happens when $|x| = 4$. When $x = 4$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} 4^n}{4^n} = \sum_{n=0}^{\infty} (-1)^{n+1},$$

which diverges. When $x = -4$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(-4)^n}{4^n} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(-1)^n 4^n}{4^n} = \sum_{n=0}^{\infty} (-1)^{2n+1} = \sum_{n=0}^{\infty} (-1),$$

which also diverges. Therefore, the interval of convergence is

$$(-4, 4).$$