Math 113 Final Exam Practice Problem Solutions

1. What are the domain and range of the function

$$f(x) = \frac{\ln x}{\sqrt{x}}?$$

Answer: $\sqrt{x}$ is only defined for $x \geq 0$, and $\ln x$ is only defined for $x > 0$. Hence, the domain of the function is $x > 0$. Notice that

$$\lim_{x \to 0^+} \frac{\ln x}{\sqrt{x}} = -\infty,$$

since $\sqrt{x} \to 0^+$ as $x \to 0^+$. Now, we can evaluate

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$$

using L'Hôpital's Rule; it is equal to

$$\lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \to \infty} \frac{2\sqrt{x}}{x} = \lim_{x \to \infty} \frac{2}{\sqrt{x}} = 0.$$

Therefore, f will have some maximum value; to figure out what it is, take

$$f'(x) = \frac{\sqrt{x}\frac{1}{x} - \ln x \frac{1}{2\sqrt{x}}}{x} = \frac{\frac{2}{2\sqrt{x}} - \frac{\ln x}{2\sqrt{x}}}{x} = \frac{2 - \ln x}{2x^{3/2}}.$$

Then $f'(x) = 0$ when

$$0 = 2 - \ln x,$$

meaning that $\ln x = 2$, or $x = e^2$. Notice that $f'(x)$ changes sign from positive to negative at $x = e^2$, so the maximum of f occurs here. Since

$$f(e^2) = \frac{\ln e^2}{\sqrt{e^2}} = \frac{2}{e},$$

we see that the range of f is

$$\left(-\infty, \frac{2}{e}\right].$$

2. Find the inverse of the function $f(x) = 1000(1 + 0.07)^x$.

Answer: To find the inverse, switch the roles of x and y, then solve for y:

$$x = 1000(1.07)^y;$$

taking the natural log of both sides, we see that

$$\ln x = \ln(1000(1.07)^y) = \ln 1000 + \ln(1.07^y) = \ln 1000 + y \ln 1.07.$$

Therefore,

$$y \ln 1.07 = \ln x - \ln 1000.$$

Hence,

$$y = \frac{\ln x - \ln 1000}{\ln 1.07}.$$

3. Find the point on the graph of $y = e^{3x}$ at which the tangent line passes through the origin.

Answer: Let $f(x) = e^{3x}$. Since
$$f'(x) = 3e^{3x},$$
the tangent line to e^{3x} at the point $x = a$ has slope $3e^{3a}$; hence, using the point-slope formula, it is given by
$$y - e^{3a} = 3e^{3a}(x - a) = 3e^{3a}x - 3ae^{3a}.$$
In other words, the tangent line to the curve at $x = a$ is
$$y = 3e^{3a}x - 3ae^{3a} + e^{3a}$$
or
$$y = e^{3a}(3x - 3a + 1).$$
This passes through the origin if we get equality when we substitute 0 for both x and y, so it must be the case that
$$0 = e^{3a}(0 - 3a + 1) = e^{3a}(1 - 3a).$$
Since $e^{3a} \neq 0$, this means that $1 - 3a = 0$, or $a = 1/3$. Therefore, since
$$f(1/3) = e^{3 \cdot 1/3} = e,$$
the point whose tangent line passes through the origin is
$$\left(\frac{1}{3}, e\right).$$

4. Find the equation of the tangent line to the curve
$$xy^3 - x^2y = 6$$
at the point $(3, 2)$.

Answer: Differentiating both sides with respect to x yields
$$y^3 + 3xy^2\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} = 0.$$
Therefore,
$$\frac{dy}{dx}\left(3xy^2 - x^2\right) = 2xy - y^3.$$
Thus,
$$\frac{dy}{dx} = \frac{2xy - y^3}{3xy^2 - x^2}.$$
Plugging in $(3, 2)$, we see that the slope of the tangent line is
$$\frac{2(3)(2) - 2^3}{3(3)(2)^2 - 3^2} = \frac{12 - 8}{36 - 9} = \frac{4}{25}.$$
Thus, using the point-slope formula, the equation of the tangent line is
$$y - 2 = \frac{4}{25}(x - 3) = \frac{4}{25}x - \frac{12}{25},$$
or, equivalently,
$$y = \frac{4}{25}x + \frac{38}{25}.$$

5. Use an appropriate linearization to approximate $\sqrt{96}$.

   Answer: Let $f(x) = \sqrt{x}$. Then I will approximate $\sqrt{96}$ using the linearization of f at $a = 100$. To do so, first take

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

   Then the linearization is

$$L(x) = f(100) + f'(100)(x - 100) = 10 + \frac{1}{20}(x - 100) = 10 + \frac{x}{20} - 5 = \frac{x}{20} + 5.$$

   Therefore,

$$\sqrt{96} = f(96) \approx L(96) = 5 + \frac{96}{20} = 5 + \frac{48}{10} = \frac{98}{10} = 9.8.$$

   So we approximate $\sqrt{96}$ by 9.8.

6. Determine the absolute maximum and minimum values of the function

$$f(x) = \frac{x}{1 + x^2}.$$

   Answer: Notice that f is defined for all x. Also,

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x}{1 + x^2} = 0,$$

   so f doesn't go off to infinity.

   Now, to find the critical points, compute

$$f'(x) = \frac{(1 + x^2)(1) - x(2x)}{(1 + x^2)^2} = \frac{1 - x^2}{(1 + x^2)^2},$$

   which equals zero precisely when $x^2 = 1$, or $x = \pm 1$. Thus, we just need to evaluate f at the critical points:

$$f(1) = \frac{1}{2}$$
$$f(-1) = -\frac{1}{2}$$

   Since f limits to 0 in both directions, we see that these are the absolute maximum and absolute minimum values of the function.

7. A specialty publisher has typically sold trade paperbacks for \$15, averaging 300 sales per week. The publisher has found that increasing the price by 50 cents reduces sales by 10 per week. If the books cost \$10 each to make, what price should the publisher charge to maximize profit?

   Answer: First, we need to figure out the demand function $p(x)$. This is a linear function with slope given by the marginal demand

$$\frac{\Delta \text{ price}}{\Delta \text{ buyers}} = \frac{0.5}{-10} = -\frac{1}{20}.$$

   The line of slope $-1/20$ going through $(300, 15)$ is

$$y - 15 = -\frac{1}{20}(x - 300) = -\frac{x}{20} + 15,$$

   or, equivalently, $y = -x/20 + 30$. Therefore,

$$p(x) = -\frac{x}{20} + 30.$$

In turn, this means the publisher's revenue is

$$R(x) = xp(x) = -\frac{x^2}{20} + 30x.$$

Also, the costs are

$$C(x) = 10x.$$

Therefore, the publisher's profit is

$$P(x) = R(x) - C(x) = -\frac{x^2}{20} + 30x - 10x = -\frac{x^2}{20} + 20x.$$

To maximize this, find the critical points:

$$P'(x) = -\frac{x}{10} + 20,$$

which equals zero when $x = 200$. Since $P''(x) = -1/10 < 0$, this critical point must be the maximum (by the second derivative test). Therefore, the publisher should charge

$$p(200) = -\frac{200}{20} + 30 = -10 + 30 = 20$$

in order to maximize profits.

8. Water is draining from a conical tank at the rate of 18 cubic feet per minute. The tank has a height of 10 feet and the radius at the top is 5 feet. How fast (in feet per minute) is the water level changing when the depth is 6 feet? (Note: the volume of a cone of radius r and height h is $\frac{\pi r^2 h}{3}$.)

   Answer: If h is the height of the top of the water in the cone and r is the radius of the top of the water, then

   $$\frac{r}{5} = \frac{h}{10},$$

   so $r = h/2$. Now, the volume of water in the tank is

   $$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi (h/2)^2 h = \frac{\pi}{12}h^3.$$

   In turn, this means that

   $$\frac{dV}{dt} = \frac{\pi}{12}3h^2\frac{dh}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}.$$

   Since $\frac{dV}{dt} = 18$, this means that

   $$18 = \frac{\pi}{4}h^2\frac{dh}{dt},$$

   or

   $$\frac{dh}{dt} = \frac{72}{\pi h^2}.$$

   Thus, when $h = 6$, the water level is changing at the rate

   $$\frac{dh}{dt} = \frac{72}{36\pi} = \frac{2}{\pi}.$$

9. Find the inflection points for the function

   $$f(x) = 8x + 3 - 2\sin x, \quad 0 < x < 3\pi.$$

Answer: Notice that

$$f'(x) = 8 - 2\cos x$$

and

$$f''(x) = 2\sin x.$$

Now, $\sin x$ changes from positive to negative at $x = \pi$ and from negative to positive at $x = 2\pi$. Since

$$f(\pi) = 8\pi + 3 - 2\sin\pi = 8\pi + 3$$
$$f(2\pi) = 8(2\pi) = 3 - 2\sin 2\pi = 16\pi + 3$$

the inflection points for f between 0 and 3π are

$$(\pi, 8\pi + 3), \quad (2\pi, 16\pi + 3).$$

10. Consider a bacteria culture that starts with a single, isolated bacterium. If the rate of change of the population of the culture is proportional to its size and if there are 100 bacteria after 1 hour, how many bacteria should we expect to see after 2 hours? [*Hint*: your answer should be a simple, recognizable number]

    Answer: Since the culture starts with a single bacterium, the population is modeled by

    $$P(t) = P_0 e^{kt} = 1 \cdot e^{kt} = e^{kt}.$$

    Now,

    $$100 = P(1) = e^{k(1)} = e^k,$$

    so $k = \ln 100$. Therefore, after 2 hours, there should be

    $$P(2) = e^{k(2)} = e^{2\ln 100} = e^{\ln 100^2} = 100^2 = 10,000$$

    bacteria in the culture.

11. Evaluate the limit

    $$\lim_{x\to 0^+} x^2 \csc^2 x.$$

    Answer: Re-write the limit as

    $$\lim_{x\to 0^+} \frac{x^2}{\sin^2 x}.$$

    Since both numerator and denominator go to zero, we can use L'Hôpital's Rule, so this limit equals

    $$\lim_{x\to 0^+} \frac{2x}{2\sin x\cos x}.$$

    Again, both numerator and denominator go to zero, so apply L'Hôpital's Rule again to get:

    $$\lim_{x\to 0^+} \frac{2}{2\cos^2 x - 2\sin^2 x} = \frac{2}{2} = 1.$$

12. Let $f(x) = x^{\cos x}$. What is $f'(\pi/2)$?

    Answer: I will use logarithmic differentiation to find $f'(x)$. To that end, let $y = f(x) = x^{\cos x}$. Then

    $$\ln y = \ln(x^{\cos x}) = \cos x \ln x.$$

    Differentiating both sides,

    $$\frac{1}{y}\frac{dy}{dx} = \cos x \frac{1}{x} - \sin x \ln x = \frac{\cos x}{x} - \sin x \ln x.$$

Therefore,

$$f'(x) = \frac{dy}{dx} = y\left(\frac{\cos x}{x} - \sin x \ln x\right) = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right).$$

Hence,

$$f'(\pi/2) = (\pi/2)^{\cos \pi/2}\left(\frac{\cos \pi/2}{\pi/2} - \sin \pi/2 \ln(\pi/2)\right) = (\pi/2)^0(0 - 1 \cdot \ln(\pi/2)) = -\ln(\pi/2).$$

13. Given that

$$f'(t) = 2t - 3\sin t, \quad f(0) = 5,$$

find f.

Answer: We know that

$$f(t) = \int f'(t)dt = \int (2t - 3\sin t)dt = t^2 + 3\cos t + C.$$

Now, since

$$5 = f(0) = 0^2 + 3\cos 0 + C = 3 + C,$$

we see that $C = 2$, so

$$f(t) = t^2 + 3\cos t + 2.$$

14. Find the absolute minimum value of the function

$$f(x) = \frac{e^x}{x}$$

for $x > 0$.

Answer: Notice that

$$\lim_{x \to 0^+} \frac{e^x}{x} = +\infty$$

and

$$\lim_{x \to \infty} \frac{e^x}{x} = \lim_{x \to \infty} \frac{e^x}{1} = \infty$$

by L'Hôpital's Rule. Therefore, we should expect the absolute minimum to occur at a critical point. To find the critical points, take the derivative:

$$f'(x) = \frac{xe^x - e^x}{x^2} = e^x \frac{x-1}{x^2}.$$

This is zero only when $x - 1 = 0$, meaning that f has a single critical point at $x = 1$. Just to double-check that this is indeed the minimum, note that f changes sign from negative to positive at $x = 1$, so, by the first derivative test, f has its minimum at $x = 1$. The minimum value of f is, thus,

$$f(1) = \frac{e^1}{1} = e.$$

15. If $\int_0^6 f(x)dx = 10$ and $\int_0^4 f(x)dx = 7$, find $\int_4^6 f(x)dx$.

Answer: Notice that

$$\int_4^6 f(x)dx = \int_0^6 f(x)dx - \int_0^4 f(x)dx = 10 - 7 = 3.$$

16. Evaluate the definite integral

$$\int_{\pi/6}^{\pi/4} \sin t dt.$$

Answer: Since $-\cos t$ is an antiderivative of $\sin t$, the Fundamental Theorem of Calculus tells us that

$$\int_{\pi/6}^{\pi/4} \sin t dt = \Big[-\cos t\Big]_{\pi/6}^{\pi/4} = -\cos(\pi/4) - (-\cos(\pi/6)) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}-\sqrt{2}}{2}.$$

17. Evaluate the integral

$$\int \sec 3t \tan 3t dt.$$

Answer: It's easy to check that $\frac{\sec 3t}{3}$ is *an* antiderivative for $\sec 3t \tan 3t$, so

$$\int \sec 3t \tan 3t dt = \frac{\sec 3t}{3} + C.$$

18. Evaluate the definite integral

$$\int_1^4 \frac{2\sqrt{x} + 4x^2}{x} dx$$

Answer: Re-write the integral as

$$\int_1^4 \left(\frac{2\sqrt{x}}{x} + \frac{4x^2}{x}\right) dx = \int_1^4 \frac{2\sqrt{x}}{x} dx + \int_1^4 \frac{4x^2}{x} dx = \int_1^4 \frac{2}{\sqrt{x}} dx + \int_1^4 4x dx.$$

Now,

$$\int_1^4 \frac{2}{\sqrt{x}} dx = \int_1^4 2x^{-1/2} dx = \left[\frac{2x^{1/2}}{1/2}\right]_1^4 = \left[4\sqrt{x}\right]_1^4 = 8 - 4 = 4.$$

On the other hand,

$$\int_1^4 4x dx = \left[2x^2\right]_1^4 = 32 - 2 = 30.$$

Therefore,

$$\int_1^4 \frac{2\sqrt{x} + 4x^2}{x} dx = \int_1^4 \frac{2}{\sqrt{x}} dx + \int_1^4 4x dx = 4 + 30 = 34.$$

19. Suppose the velocity of a particle is given by

$$v(t) = 6t^2 - 4t.$$

What is the displacement of the particle from 0 to 2?

Answer: The displacement is given by

$$s(2) - s(0).$$

Since $s'(t) = v(t)$, the Fundamental Theorem tells us that

$$s(2) - s(0) = \int_0^2 s'(t) dt = \int_0^2 v(t) dt = \int_0^2 (6t^2 - 4t) dt = \left[2t^3 - 2t^2\right]_0^2 = (16 - 8) - (0 - 0) = 8.$$

Therefore, the displacement is 8 units.

20. Suppose that

$$\int_0^{x^2} f(t)dt = \sqrt{x^2+1} - 1.$$

What is $f(2)$?

Answer: Let $g(x) = \sqrt{x^2+1} - 1$. Then,

$$g'(x) = \frac{d}{dx}\left(\int_0^{x^2} f(t)dt\right) = \frac{d}{du}\left(\int_0^{u} f(t)dt\right)\frac{du}{dx}$$

where $u = x^2$, using the Chain Rule.

Therefore, by the first part of the Fundamental Theorem,

$$g'(x) = f(u) \cdot 2x = 2xf(x^2).$$

In other words,

$$f(x^2) = \frac{g'(x)}{2x}.$$

Now, we know that $g(x) = \sqrt{x^2+1} - 1$, so

$$g'(x) = \frac{1}{2\sqrt{x^2+1}}(2x) = \frac{x}{\sqrt{x^2+1}}.$$

Hence,

$$f(2) = \frac{g'(\sqrt{2})}{2\sqrt{2}} = \frac{\frac{\sqrt{2}}{\sqrt{3}}}{2\sqrt{2}} = \frac{1}{2\sqrt{3}}.$$