

Math 113 Exam #2 Practice Problem Solutions

1. Let $u(x) = \sqrt{f(x)}$ and suppose $f(3) = 1$, $f'(3) = 8$, and $f''(3) = -2$. What is the value of $u''(3)$?

Answer: Using the Chain Rule,

$$u'(x) = \frac{1}{2}(f(x))^{-1/2}f'(x).$$

Hence, using the Chain Rule and the Product Rule,

$$u''(x) = \frac{1}{2} \left(-\frac{1}{2}(f(x))^{-3/2}f'(x)f'(x) + (f(x))^{-1/2}f''(x) \right) = -\frac{(f'(x))^2}{4(f(x))^{3/2}} + \frac{f''(x)}{2\sqrt{f(x)}}.$$

Therefore,

$$u''(3) = -\frac{(f'(3))^2}{4(f(3))^{3/2}} + \frac{f''(3)}{\sqrt{f(3)}} = -\frac{8^2}{4(1)^{3/2}} + \frac{-2}{2\sqrt{1}} = -17.$$

2. Let $f(x) = \frac{1}{2} \sin(x^2) \cos(x^2)$. What is $f' \left(\sqrt{\frac{5\pi}{6}} \right)$?

Answer: Using the Product and Chain Rules,

$$f'(x) = \frac{1}{2} (\cos(x^2) \cdot 2x \cdot \cos(x^2) + \sin(x^2) \cdot 2x \cdot (-\sin(x^2))) = x \cos^2(x^2) - x \sin^2(x^2) = x (\cos^2(x^2) - \sin^2(x^2)).$$

Therefore,

$$\begin{aligned} f' \left(\sqrt{\frac{5\pi}{6}} \right) &= \sqrt{\frac{5\pi}{6}} \left(\cos^2 \left(\frac{5\pi}{6} \right) - \sin^2 \left(\frac{5\pi}{6} \right) \right) \\ &= \sqrt{\frac{5\pi}{6}} \left(\left(-\frac{\sqrt{3}}{2} \right)^2 - \left(\frac{1}{2} \right)^2 \right) \\ &= \sqrt{\frac{5\pi}{6}} \left(\frac{3}{4} - \frac{1}{4} \right) \\ &= \frac{1}{2} \sqrt{\frac{5\pi}{6}}. \end{aligned}$$

3. Suppose you've leaned a 10 foot ladder against a vertical wall, but haven't properly secured the bottom of the ladder. Before you can climb onto the ladder, the bottom starts to slide away from the wall at 3 ft/sec. How fast does the top of the ladder slide down the wall when the bottom is 6 feet from the wall?

Answer: The ladder, the floor and the wall form a right triangle whose hypotenuse is the ladder. Hence, if the ladder touches the wall h feet above the floor and touches the floor f feet away from the wall, we have that

$$10^2 = f^2 + h^2.$$

Also, we know that, at the time of interest, $f = 6$ and $\frac{df}{dt} = 3$. Our goal is to find $\frac{dh}{dt}$ at this time.

To do so, first note that, at this time,

$$10^2 = 6^2 + h^2,$$

So

$$h = \sqrt{100 - 36} = \sqrt{64} = 8.$$

Now, differentiate both sides of the expression $10^2 = f^2 + h^2$ with respect to t :

$$0 = 2f \frac{df}{dt} + 2h \frac{dh}{dt}.$$

Therefore, at the moment of interest,

$$0 = 2(6)(3) + 2(8)\frac{dh}{dt}.$$

Hence,

$$\frac{dh}{dt} = -\frac{36}{16} = -\frac{9}{4},$$

so the top of the ladder is sliding down the wall at $\frac{9}{4}$ ft/sec at this instant.

4. Suppose a cup of boiling water is left to cool on the counter. If the room temperature is 20°C and the water has a temperature of 60°C after 10 minutes, what will be the temperature of the water after half an hour? (Remember that water boils at 100°C)

Answer: Newton's Law of Cooling says that, if $T(t)$ is the temperature of the water and T_s is the temperature of the room,

$$\frac{dT}{dt} = k(T(t) - T_s).$$

We've seen that solutions of this equation take the form

$$T(t) = T_s + Ce^{kt}.$$

In this case, $T_s = 20$, so the equation for $T(t)$ is $20 + Ce^{kt}$. Now, plug $t = 0$ into this equation:

$$100 = 20 + Ce^{k \cdot 0} = 20 + C,$$

so $C = 80$ and we have that

$$T(t) = 20 + 80e^{kt}.$$

Plugging $t = 10$ into this equation yields

$$60 = 20 + 80e^{10k}.$$

Therefore,

$$80e^{10k} = 40,$$

so

$$e^{10k} = \frac{1}{2}.$$

Taking the natural log of both sides, we see that

$$10k = \ln\left(\frac{1}{2}\right),$$

so

$$k = \frac{\ln\left(\frac{1}{2}\right)}{10}.$$

Therefore, after 30 minutes, the temperature of the water is given by

$$20 + 80e^{\frac{\ln\left(\frac{1}{2}\right)}{10} \cdot 30} = 20 + 80e^{3\ln\left(\frac{1}{2}\right)}.$$

Since $3\ln\left(\frac{1}{2}\right) = \ln\left(\frac{1}{2}\right)^3 = \ln\left(\frac{1}{8}\right)$, we see that the temperature of the water after 30 minutes is

$$20 + 80e^{\ln\left(\frac{1}{8}\right)} = 20 + 80 \cdot \frac{1}{8} = 30^\circ\text{C}.$$

5. What is the tangent line to $x^3 + y^3 = 6xy$ at $(3, 3)$?

Answer: Differentiating both sides of the equation with respect to x yields

$$3x^2 + 3y^2 \frac{dy}{dx} = 6 \left(y + x \frac{dy}{dx} \right).$$

Re-arranging gives that

$$(3y^2 - 6x) \frac{dy}{dx} = 6y - 3x^2,$$

so

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}.$$

Plugging in $(x, y) = (3, 3)$, we see that the slope of the tangent line at this point is

$$\frac{6(3) - 3(3)^2}{3(3)^2 - 6(3)} = \frac{18 - 27}{27 - 18} = \frac{-9}{9} = -1.$$

Therefore, using the point-slope formula, the equation of the tangent line is

$$y - 3 = -1(x - 3),$$

or

$$y = -x + 6.$$

6. Suppose $y = x^{1/x}$. What is $\frac{dy}{dx}$?

Answer: We will find the derivative using logarithmic differentiation. Thus, we first take the natural log of both sides:

$$\ln y = \ln(x^{1/x}) = \frac{1}{x} \ln x.$$

Now, differentiating both sides with respect to x , we see that

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} \cdot \frac{1}{x} - \frac{1}{x^2} \ln x = \frac{1 - \ln x}{x^2}$$

Multiplying both sides by y gives us that

$$\frac{dy}{dx} = y \frac{1 - \ln x}{x^2}$$

so, plugging in $y = x^{1/x}$ we see that

$$\frac{dy}{dx} = x^{1/x} \frac{1 - \ln x}{x^2}.$$

7. Use a linearization of an appropriate function to estimate $\ln(0.9)$.

Answer: We will approximate $\ln(0.9)$ using the linearization of $f(x) = \ln x$ at 1. This linearization is given by

$$L(x) = f(1) + f'(1)(x - 1).$$

Since $f'(x) = \frac{1}{x}$, we have that

$$f(1) = \ln 1 = 0$$

$$f'(1) = \frac{1}{1} = 1.$$

Plugging these values into the expression for $L(x)$ yields

$$L(x) = 0 + (x - 1) = x - 1.$$

Therefore,

$$\ln(0.9) \approx L(0.9) = 0.9 - 1 = -0.1.$$