Math 113 Exam #1 Practice Problems

1. Find the vertical asymptotes (if any) of the functions

$$g(x) = 1 + \frac{2}{x}, \qquad h(x) = \frac{4x}{4 - x^2}$$

What are the domains of g and h?

Answer: The function g has a vertical asymptote at $x = 0$. The function h has vertical asymptotes when $4 - x^2 = 0$, so they're at $x = -2$ and $x = 2$.

2. Evaluate

$$\text{(a)} \lim_{x\to 2} \frac{x^2 - 4}{x^2 - 5x + 6} \qquad \text{(b)} \lim_{x\to -2} \frac{|x+2|}{x+2} \qquad \text{(c)} \lim_{x\to\infty} \frac{4x^3 + 2x - 4}{4x^2 - 5x + 6x^3}$$

(a) We can factor the numerator as

$$x^2 - 4 = (x+2)(x-2)$$

and the denominator as

$$x^2 - 5x + 6 = (x-2)(x-3).$$

Therefore,

$$\lim_{x\to 2} \frac{x^2 - 4}{x^2 - 5x + 6} = \lim_{x\to 2} \frac{(x+2)(x-2)}{(x-2)(x-3)} = \lim_{x\to 2} \frac{x+2}{x-3} = \frac{4}{-1} = -4.$$

(b) When $x < -2$, the quantity $x + 2$ is negative, so

$$|x+2| = -(x+2).$$

Hence,

$$\lim_{x\to -2^-} \frac{|x+2|}{x+2} = \lim_{x\to -2^-} \frac{-(x+2)}{x+2} = -1.$$

On the other hand, when $x > -2$, the quantity $x + 2$ is positive, so

$$|x+2| = x+2.$$

Therefore,

$$\lim_{x\to -2^+} \frac{|x+2|}{x+2} = \lim_{x\to -2^+} \frac{x+2}{x+2} = 1.$$

Since the limits from the left and right don't agree,

$$\lim_{x\to -2} \frac{|x+2|}{x+2}$$

does not exist.

(c) Dividing numerator and denominator by x^3, we get that

$$\begin{aligned}
\lim_{x\to\infty} \frac{4x^3 + 2x - 4}{4x^2 - 5x + 6x^3} &= \lim_{x\to\infty} \frac{\frac{1}{x^3}\left(4x^3 + 2x - 4\right)}{\frac{1}{x^3}\left(4x^2 - 5x + 6x^3\right)} \\
&= \lim_{x\to\infty} \frac{4 + \frac{2}{x^2} - \frac{4}{x^3}}{\frac{4}{x} - \frac{5}{x^2} + 6} \\
&= \frac{4}{6} \\
&= \frac{2}{3}.
\end{aligned}$$

3. Evaluate

$$\lim_{x\to 6} \frac{x^2-36}{3x^2-16x-12}$$

Answer: The numerator factors as

$$\frac{x^2-36}{=}(x+6)(x-6),$$

while the denominator factors as

$$3x^2-16x-12=(3x+2)(x-6).$$

Therefore,

$$\lim_{x\to 6} \frac{x^2-36}{3x^2-16x-12} = \lim_{x\to 6} \frac{(x+6)(x-6)}{3x+2)(x-6)} = \lim_{x\to 6} \frac{x+6}{3x+2} = \frac{12}{20} = \frac{3}{5}$$

4. Evaluate

$$\lim_{x\to\infty} \frac{\sqrt[3]{x^2-3x+29034}}{7x-9999}$$

Answer: Dividing numerator and denominator by x, we see that

$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt[3]{x^2-3x+29034}}{7x-9999} &= \lim_{x\to\infty} \frac{\frac{1}{x}\sqrt[3]{x^2-3x+29034}}{\frac{1}{x}(7x-9999)} \\ &= \lim_{x\to\infty} \frac{\sqrt[3]{\frac{1}{x^3}\left(x^2-3x+29034\right)}}{7-\frac{9999}{x}} \\ &= \lim_{x\to\infty} \frac{\sqrt[3]{\frac{1}{x}-\frac{3}{x^2}+\frac{29034}{x^3}}}{7-\frac{9999}{x}} \\ &= 0.\end{aligned}$$

5. Let

$$f(x) = \begin{cases} cx^2-3 & \text{if } x \le 2 \\ cx+2 & \text{if } x > 2 \end{cases}$$

f is continuous provided c equals what value?

Answer: Since both cx^2-3 and $cx+2$ are polynomials, they're continuous everywhere, meaning that $f(x)$ is continuous everywhere except possibly at $x=2$. In order for f to be continuous at 2, it must be the case that $f(2)=\lim_{x\to 2} f(x)$. Now,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \left(cx^2-3\right) = c(2)^2-3 = 4c-3,$$

which is also the value of $f(2)$. On the other hand,

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (cx+2) = c(2)+2 = 2c+2.$$

f will be continuous when these two one-sided limits are equal, meaning when

$$4c-3=2c+2.$$

Solving for c, we see that f is continuous when

$$c=\frac{2}{5}.$$

6. Is the function f defined below continuous? If not, where is it discontinuous?

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ 3 - x & \text{if } 0 \leq x < 3 \\ (3-x)^2 & \text{if } x \geq 3 \end{cases}$$

Answer: Since each of the three pieces of f is continuous, the only possible discontinuities of f occur where it switches from one piece to another, namely at $x = 0$ and $x = 3$. For $x \to 3$, both $x - 3$ and $(x-3)^2$ go to zero, so f is continuous at $x = 3$. On the other hand,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \sqrt{-x} = 0,$$

whereas

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (3 - x) = 3,$$

so f is discontinuous at $x = 0$.

7. Let $f(x)$ be continuous on the closed interval $[-3, 6]$. If $f(-3) = -1$ and $f(6) = 3$, then which of the following must be true?

(a) $f(0) = 0$

(b) $f'(c) = \frac{4}{9}$ for at least one c between -3 and 6

(c) $-1 \leq f(x) \leq 3$ for all x between -3 and 6.

(d) $f(c) = 1$ for at least one c between -3 and 6.

(e) $f(c) = 0$ for at least one c between -1 and 3.

Answer: The only one of these statements which is necessarily true is (d): since 1 is between $f(-3) = -1$ and $f(6) = 3$, the Intermediate Value Theorem guarantees that there is some c between -3 and 6 such that $f(c) = 1$.

8. Find the one-sided limit

$$\lim_{x \to -1^-} \frac{x - 1}{x^4 - 1}$$

Answer: Notice that, as $x \to -1$, the numerator goes to -2, while the denominator goes to zero. Hence, we would expect the limit to be infinite. However, it could be either $-\infty$ or $+\infty$, so we need to check the sign of the denominator.

When $x < -1$, the quantity $x^4 > 1$, so

$$x^4 - 1 > 0.$$

Therefore, in the one-sided limit, the denominator is always positive. Since the numerator goes to -2, which is negative, the one-sided limit

$$\lim_{x \to -1^-} \frac{x - 1}{x^4 - 1} = -\infty.$$

9. Let

$$f(x) = x^3 + 2x^2 + 1.$$

Is f differentiable at -2? If so, what is $f'(-2)$?

Answer: f is differentiable at -2 if $f'(-2)$ exists. By definition,

$$\begin{aligned}
f'(-2) &= \lim_{h\to 0} \frac{f(-2+h)-f(-2)}{h} \\
&= \lim_{h\to 0} \frac{\left[(-2+h)^3+2(-2+h)^2+1\right]-\left[(-2)^3+2(-2)^2+1\right]}{h} \\
&= \lim_{h\to 0} \frac{\left[(-2)^3+3(-2)^2h+3(-2)h^2+h^3+2((-2)^2-4h+h^2)+1\right]-\left[(-2)^3+2(-2)^2+1\right]}{h}
\end{aligned}$$

Canceling the terms without h's in them and simplifying yields

$$f'(-2) = \lim_{h\to 0} \frac{4h-4h^2+h^3}{h} = \lim_{h\to 0}(4-4h+h^2) = 4,$$

so f is differentiable at -2 and $f'(-2) = 4$.

10. Let

$$f(x) = |x-2|.$$

Is f differentiable at 2? If so, what is $f'(2)$?

Answer: f is not differentiable at 2. To see this, note that, if $f'(2)$ exists, then it should be equal to

$$\lim_{h\to 0} \frac{f(2+h)-f(2)}{h}.$$

To see that this limit does not exist, I will examine the two one-sided limits:

$$\begin{aligned}
\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0^-} \frac{|(2+h)-2|-|2-2|}{h} \\
&= \lim_{h\to 0^-} \frac{|h|-0}{h} \\
&= \lim_{h\to 0^-} \frac{|h|}{h} \\
&= \lim_{h\to 0^-} \frac{-h}{h} \\
&= -1
\end{aligned}$$

since $|h| = -h$ when $h < 0$.

On the other hand,

$$\begin{aligned}
\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0^+} \frac{|(2+h)-2|-|2-2|}{h} \\
&= \lim_{h\to 0^+} \frac{|h|}{h} \\
&= \lim_{h\to 0^+} \frac{h}{h} \\
&= 1
\end{aligned}$$

since $|h| = h$ when $h > 0$.

Therefore, since the two one-sided limits don't agree, the limit does not exist, so f is not differentiable at $x = 2$.