

Poincaré Duality Angles on Riemannian Manifolds with Boundary

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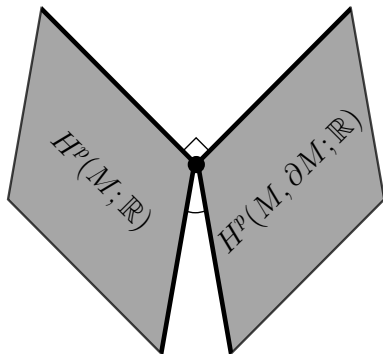
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de Rham's Theorem

Suppose M^n is a compact, oriented, smooth manifold. Then

$$H^p(M; \mathbb{R}) \cong \mathcal{C}^p(M) / \mathcal{E}^p(M),$$

where $\mathcal{C}^p(M)$ is the space of closed p -forms on M and $\mathcal{E}^p(M)$ is the space of exact p -forms.

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Kodaira called this the space of *harmonic p -fields* on M .

Hodge's Theorem

If M^n is a closed, oriented, smooth Riemannian manifold,

$$H^p(M; \mathbb{R}) \cong \mathcal{H}^p(M).$$

Hodge–Morrey–Friedrichs Decomposition

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The L^2 -orthogonal complement of the exact forms inside the space of closed forms is now:

$$\mathcal{H}_N^p(M) := \{\omega \in \Omega^p(M) : d\omega = 0, \delta\omega = 0, i^* \star \omega = 0\}.$$

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Then

$$H^p(M; \mathbb{R}) \cong \mathcal{H}_N^p(M).$$

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If $i^*\omega = 0$, then ω is said to satisfy the *Dirichlet boundary condition*.

The concrete realizations of $H^p(M; \mathbb{R})$ and $H^p(M, \partial M; \mathbb{R})$ meet only at the origin:

$$\mathcal{H}_N^p(M) \cap \mathcal{H}_D^p(M) = \{0\}$$

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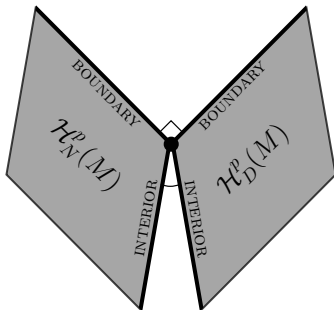
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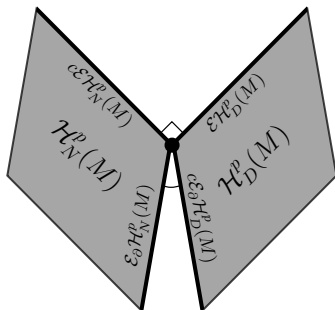
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Interior subspace of $\mathcal{H}_D^p(M)$:

$$\begin{aligned} \star \mathcal{E}_\partial \mathcal{H}_N^{n-p}(M) &= c \mathcal{E}_\partial \mathcal{H}_D^p(M) \\ &= \{\eta \in \mathcal{H}_D^p(M) : i^* \star \eta = d\psi, \psi \in \Omega^{n-p-1}(\partial M)\}. \end{aligned}$$



Definition (DeTurck–Gluck)

The *Poincaré duality angles* of the Riemannian manifold M are the principal angles between the interior subspaces.

What do the Poincaré duality angles tell you?

Guess

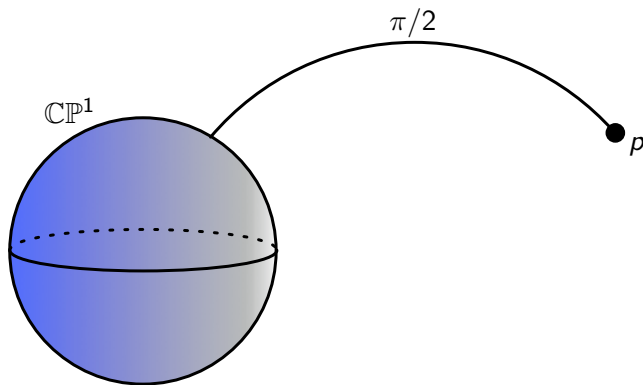
If M is “almost” closed, the Poincaré duality angles of M should be small.

Consider $\mathbb{C}\mathbb{P}^2$ with its usual Fubini-Study metric. Let $p \in \mathbb{C}\mathbb{P}^2$.
Then define

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M_r has absolute cohomology in dimensions 0 and 2.

M_r has relative cohomology in dimensions 2 and 4.

Therefore, M_r has a single Poincaré duality angle θ_r between $\mathcal{H}_N^2(M_r)$ and $\mathcal{H}_D^2(M_r)$.

Find harmonic 2-fields

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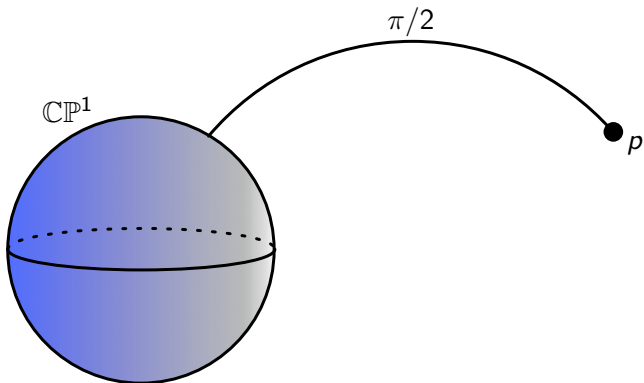
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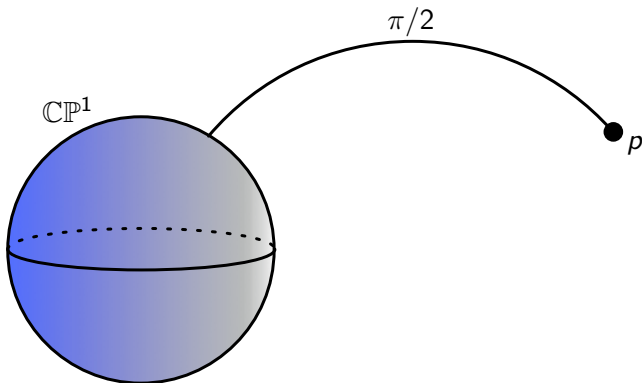
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Find closed and co-closed $SU(2)$ -invariant forms on M_r satisfying Neumann and Dirichlet boundary conditions.

Finding isometry-invariant 2-forms



Finding isometry-invariant 2-forms



The hypersurfaces at constant distance t from $\mathbb{C}P^1$ are Berger 3-spheres:

$$S^3(\cos t)_{\sin t}$$

The Poincaré duality angle for M_r

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Generalizes to $\mathbb{C}P^n - B_r(p)$.

Poincaré duality angles of Grassmannians

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Theorem

- As $r \rightarrow 0$, all the Poincaré duality angles of N_r go to zero.
- As r approaches its maximum value of $\pi/2$, all the Poincaré duality angles of N_r go to $\pi/2$.

Conjecture

If M^n is a closed Riemannian manifold and N^k is a closed submanifold of codimension ≥ 2 , the Poincaré duality angles of

$$M - \nu_r(N)$$

go to zero as $r \rightarrow 0$.

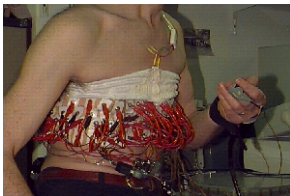
What can you learn about the topology of M from knowledge of ∂M ?

Electrical Impedance Tomography

Induce potentials on the boundary of a region and determine the conductivity inside the region by measuring the current flux through the boundary.

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The Voltage-to-Current map

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Then f extends to a potential u on M , where

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If γ is the conductivity on M , the current flux through ∂M is given by

$$(\gamma \nabla u) \cdot \nu = -\gamma \frac{\partial u}{\partial \nu}$$

The Dirichlet-to-Neumann map

The map $\Lambda_{\text{cl}} : C^\infty(\partial M) \rightarrow C^\infty(\partial M)$ defined by

$$f \mapsto \frac{\partial u}{\partial \nu}$$

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Theorem (Lee-Uhlmann)

If M^n is a compact, analytic Riemannian manifold with boundary, then M is determined up to isometry by Λ_{cl} .

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$$\Lambda : \Omega^p(\partial M) \rightarrow \Omega^{n-p-1}(\partial M)$$

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$$\Delta\omega = 0, \quad i^*\omega = \varphi, \quad i^*\delta\omega = 0.$$

Generalization to differential forms

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If $f \in \Omega^0(\partial M)$,

$$\Lambda f = i^* \star du = \frac{\partial u}{\partial \nu} \text{dvol}_{\partial M} = (\Lambda_{\text{cl}} f) \text{dvol}_{\partial M}$$

Theorem (Belishev–Sharafutdinov)

The data $(\partial M, \Lambda)$ completely determines the cohomology groups of M .

Connection to Poincaré duality angles

Define the *Hilbert transform* $T := d\Lambda^{-1}$.

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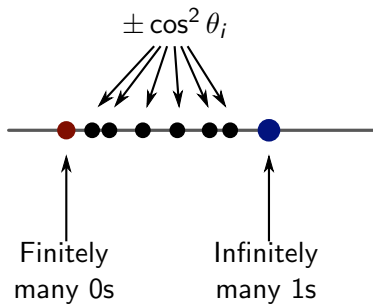
Theorem

If $\theta_1, \dots, \theta_k$ are the Poincaré duality angles of M in dimension p , then the quantities

$$(-1)^{np+n+p} \cos^2 \theta_i$$

are the non-zero eigenvalues of an appropriate restriction of T^2 .

The Spectrum of T^2

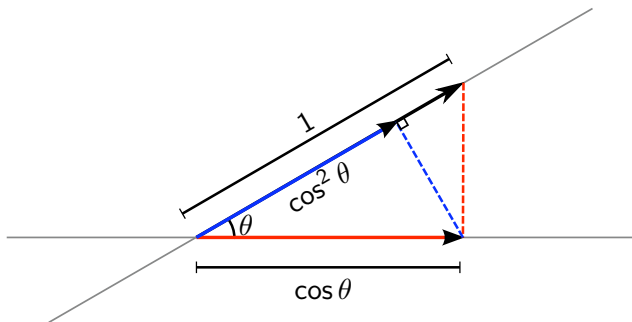


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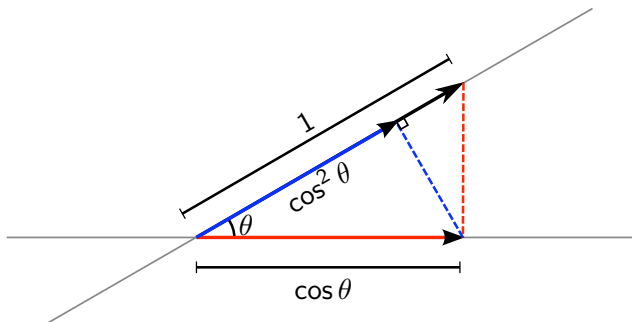
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The Hilbert transform T recaptures the orthogonal projection $\mathcal{H}_N^p(M) \rightarrow \mathcal{H}_D^p(M)$.

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Theorem

The mixed cup product

$$\cup : H^p(M; \mathbb{R}) \times H^q(M, \partial M; \mathbb{R}) \rightarrow H^{p+q}(M, \partial M; \mathbb{R})$$

is completely determined by the data $(\partial M, \Lambda)$ when the relative class is restricted to come from the boundary subspace.

Thanks!