

# Poincaré Duality Angles on Riemannian Manifolds with Boundary

Clayton Shonkwiler

Department of Mathematics  
Haverford College

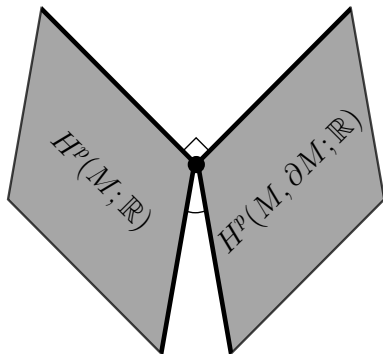
March 4, 2010

# Realizing cohomology groups as spaces of differential forms

Let  $M^n$  be a compact Riemannian manifold with non-empty boundary  $\partial M$ .

# Realizing cohomology groups as spaces of differential forms

Let  $M^n$  be a compact Riemannian manifold with non-empty boundary  $\partial M$ .



## de Rham's Theorem

*Suppose  $M^n$  is a compact, oriented, smooth manifold. Then*

$$H^p(M; \mathbb{R}) \cong \mathcal{C}^p(M) / \mathcal{E}^p(M),$$

*where  $\mathcal{C}^p(M)$  is the space of closed  $p$ -forms on  $M$  and  $\mathcal{E}^p(M)$  is the space of exact  $p$ -forms.*

If  $M$  is Riemannian, the metric induces an  $L^2$  inner product on  $\Omega^p(M)$ :

$$\langle \omega, \eta \rangle := \int_M \omega \wedge \star \eta.$$

If  $M$  is Riemannian, the metric induces an  $L^2$  inner product on  $\Omega^p(M)$ :

$$\langle \omega, \eta \rangle := \int_M \omega \wedge \star \eta.$$

When  $M$  is closed, the orthogonal complement of  $\mathcal{E}^p(M)$  inside  $\mathcal{C}^p(M)$  is

$$\mathcal{H}^p(M) := \{ \omega \in \Omega^p(M) : d\omega = 0, \delta\omega = 0 \}$$

If  $M$  is Riemannian, the metric induces an  $L^2$  inner product on  $\Omega^p(M)$ :

$$\langle \omega, \eta \rangle := \int_M \omega \wedge \star \eta.$$

When  $M$  is closed, the orthogonal complement of  $\mathcal{E}^p(M)$  inside  $\mathcal{C}^p(M)$  is

$$\mathcal{H}^p(M) := \{ \omega \in \Omega^p(M) : d\omega = 0, \delta\omega = 0 \}$$

Kodaira called this the space of *harmonic  $p$ -fields* on  $M$ .

## Hodge's Theorem

*If  $M^n$  is a closed, oriented, smooth Riemannian manifold,*

$$H^p(M; \mathbb{R}) \cong \mathcal{H}^p(M).$$

# Hodge–Morrey–Friedrichs Decomposition

Define  $i : \partial M \hookrightarrow M$  to be the natural inclusion.

# Hodge–Morrey–Friedrichs Decomposition

Define  $i : \partial M \hookrightarrow M$  to be the natural inclusion.

The  $L^2$ -orthogonal complement of the exact forms inside the space of closed forms is now:

$$\mathcal{H}_N^p(M) := \{\omega \in \Omega^p(M) : d\omega = 0, \delta\omega = 0, i^* \star \omega = 0\}.$$

$$i^* \star \omega = 0$$

Suppose  $dx_n$  corresponds to the normal direction.

If  $i^* \star \omega = 0$ , then

Suppose  $dx_n$  corresponds to the normal direction.

If  $i^* \star \omega = 0$ , then

$$\star \omega = \left( \sum f_I dx_I \right) \wedge dx_n,$$

Suppose  $dx_n$  corresponds to the normal direction.

If  $i^* \star \omega = 0$ , then

$$\star \omega = \left( \sum f_I dx_I \right) \wedge dx_n,$$

meaning that  $\omega$  has no  $dx_n$  in it.

# Hodge–Morrey–Friedrichs Decomposition

Define  $i : \partial M \hookrightarrow M$  to be the natural inclusion.

The  $L^2$ -orthogonal complement of the exact forms inside the space of closed forms is now:

$$\mathcal{H}_N^p(M) := \{\omega \in \Omega^p(M) : d\omega = 0, \delta\omega = 0, i^* \star \omega = 0\}.$$

# Hodge–Morrey–Friedrichs Decomposition

Define  $i : \partial M \hookrightarrow M$  to be the natural inclusion.

The  $L^2$ -orthogonal complement of the exact forms inside the space of closed forms is now:

$$\mathcal{H}_N^p(M) := \{\omega \in \Omega^p(M) : d\omega = 0, \delta\omega = 0, i^* \star \omega = 0\}.$$

Then

$$H^p(M; \mathbb{R}) \cong \mathcal{H}_N^p(M).$$

# Hodge–Morrey–Friedrichs Decomposition (continued)

The relative cohomology appears as

$$H^p(M, \partial M; \mathbb{R}) \cong \mathcal{H}_D^p(M).$$

# Hodge–Morrey–Friedrichs Decomposition (continued)

The relative cohomology appears as

$$H^p(M, \partial M; \mathbb{R}) \cong \mathcal{H}_D^p(M).$$

$$\mathcal{H}_D^p(M) := \{\omega \in \Omega^p(M) : d\omega = 0, \delta\omega = 0, i^*\omega = 0\}.$$

The concrete realizations of  $H^p(M; \mathbb{R})$  and  $H^p(M, \partial M; \mathbb{R})$  meet only at the origin:

$$\mathcal{H}_N^p(M) \cap \mathcal{H}_D^p(M) = \{0\}$$

The concrete realizations of  $H^p(M; \mathbb{R})$  and  $H^p(M, \partial M; \mathbb{R})$  meet only at the origin:

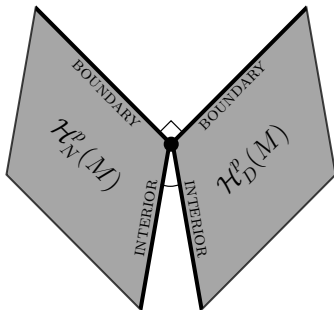
$$\mathcal{H}_N^p(M) \cap \mathcal{H}_D^p(M) = \{0\}$$

...but they are not orthogonal!

The concrete realizations of  $H^p(M; \mathbb{R})$  and  $H^p(M, \partial M; \mathbb{R})$  meet only at the origin:

$$\mathcal{H}_N^p(M) \cap \mathcal{H}_D^p(M) = \{0\}$$

...but they are not orthogonal!



Interior subspace of  $\mathcal{H}_N^p(M)$ :

$$\ker i^* \text{ where } i^* : H^p(M; \mathbb{R}) \rightarrow H^p(\partial M; \mathbb{R})$$

Interior subspace of  $\mathcal{H}_N^p(M)$ :

$$\ker i^* \text{ where } i^* : H^p(M; \mathbb{R}) \rightarrow H^p(\partial M; \mathbb{R})$$

$$\mathcal{E}_\partial \mathcal{H}_N^p(M) := \{\omega \in \mathcal{H}_N^p(M) : i^* \omega = d\varphi, \varphi \in \Omega^{p-1}(\partial M)\}.$$

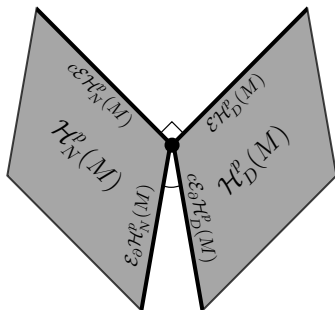
Interior subspace of  $\mathcal{H}_N^p(M)$ :

$$\ker i^* \text{ where } i^* : H^p(M; \mathbb{R}) \rightarrow H^p(\partial M; \mathbb{R})$$

$$\mathcal{E}_\partial \mathcal{H}_N^p(M) := \{\omega \in \mathcal{H}_N^p(M) : i^* \omega = d\varphi, \varphi \in \Omega^{p-1}(\partial M)\}.$$

Interior subspace of  $\mathcal{H}_D^p(M)$ :

$$\begin{aligned} \star \mathcal{E}_\partial \mathcal{H}_N^{n-p}(M) &= c \mathcal{E}_\partial \mathcal{H}_D^p(M) \\ &= \{\eta \in \mathcal{H}_D^p(M) : i^* \star \eta = d\psi, \psi \in \Omega^{n-p-1}(\partial M)\}. \end{aligned}$$



## Definition (DeTurck–Gluck)

The *Poincaré duality angles* of the Riemannian manifold  $M$  are the principal angles between the interior subspaces.

# What do the Poincaré duality angles tell you?

## Guess

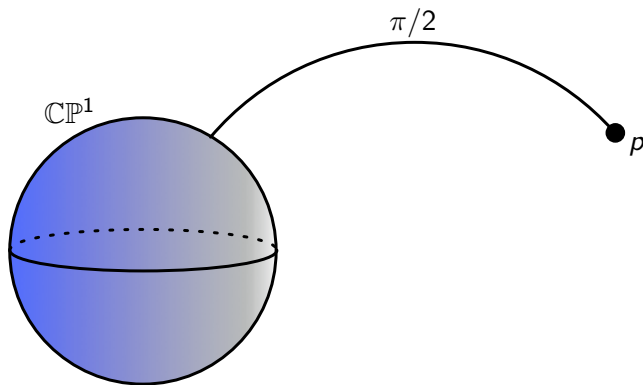
*If  $M$  is “almost” closed, the Poincaré duality angles of  $M$  should be small.*

Consider  $\mathbb{C}\mathbb{P}^2$  with its usual Fubini-Study metric. Let  $p \in \mathbb{C}\mathbb{P}^2$ .  
Then define

$$M_r := \mathbb{C}\mathbb{P}^2 - B_r(p).$$

Consider  $\mathbb{C}\mathbb{P}^2$  with its usual Fubini-Study metric. Let  $p \in \mathbb{C}\mathbb{P}^2$ .  
Then define

$$M_r := \mathbb{C}\mathbb{P}^2 - B_r(p).$$



$\partial M_r$  is a 3-sphere.

$\partial M_r$  is a 3-sphere.

$M_r$  is the  $D^2$ -bundle over  $\mathbb{C}\mathbb{P}^1$  ( $\simeq S^2(1/2)$ ) with Euler characteristic 1.

$\partial M_r$  is a 3-sphere.

$M_r$  is the  $D^2$ -bundle over  $\mathbb{C}\mathbb{P}^1$  ( $\simeq S^2(1/2)$ ) with Euler characteristic 1.

$M_r$  has absolute cohomology in dimensions 0 and 2.

$\partial M_r$  is a 3-sphere.

$M_r$  is the  $D^2$ -bundle over  $\mathbb{C}\mathbb{P}^1$  ( $\simeq S^2(1/2)$ ) with Euler characteristic 1.

$M_r$  has absolute cohomology in dimensions 0 and 2.

$M_r$  has relative cohomology in dimensions 2 and 4.

$\partial M_r$  is a 3-sphere.

$M_r$  is the  $D^2$ -bundle over  $\mathbb{C}\mathbb{P}^1$  ( $\simeq S^2(1/2)$ ) with Euler characteristic 1.

$M_r$  has absolute cohomology in dimensions 0 and 2.

$M_r$  has relative cohomology in dimensions 2 and 4.

Therefore,  $M_r$  has a single Poincaré duality angle  $\theta_r$  between  $\mathcal{H}_N^2(M_r)$  and  $\mathcal{H}_D^2(M_r)$ .

## Find harmonic 2-fields

So the goal is to find closed and co-closed 2-forms on  $M_r$  which satisfy Neumann and Dirichlet boundary conditions.

## Find harmonic 2-fields

So the goal is to find closed and co-closed 2-forms on  $M_r$  which satisfy Neumann and Dirichlet boundary conditions.

Such 2-forms must be isometry-invariant.

So the goal is to find closed and co-closed 2-forms on  $M_r$  which satisfy Neumann and Dirichlet boundary conditions.

Such 2-forms must be isometry-invariant.

$$\text{Isom}_0(M_r) = SU(2).$$

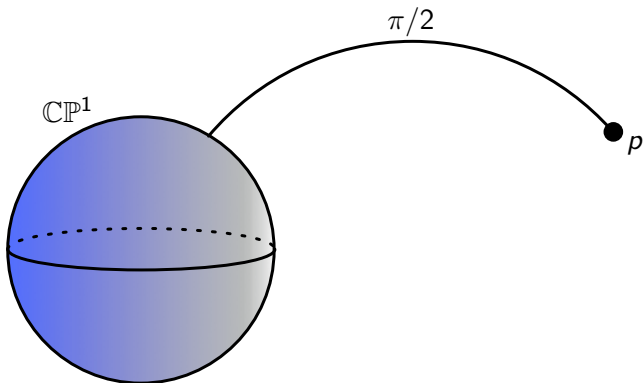
So the goal is to find closed and co-closed 2-forms on  $M_r$  which satisfy Neumann and Dirichlet boundary conditions.

Such 2-forms must be isometry-invariant.

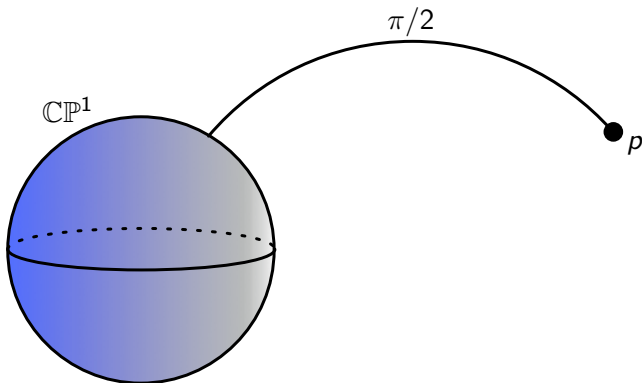
$$\text{Isom}_0(M_r) = SU(2).$$

Find closed and co-closed  $SU(2)$ -invariant forms on  $M_r$  satisfying Neumann and Dirichlet boundary conditions

# Finding isometry-invariant 2-forms



# Finding isometry-invariant 2-forms



The hypersurfaces at constant distance  $t$  from  $\mathbb{C}P^1$  are Berger 3-spheres:

$$S^3(\cos t)_{\sin t}$$

# The Poincaré duality angle for $M_r$

$$\cos \theta_r = \frac{1 - \sin^4 r}{1 + \sin^4 r}.$$

# The Poincaré duality angle for $M_r$

$$\cos \theta_r = \frac{1 - \sin^4 r}{1 + \sin^4 r}.$$

As  $r \rightarrow 0$ , the Poincaré duality angle  $\theta_r \rightarrow 0$ .

# The Poincaré duality angle for $M_r$

$$\cos \theta_r = \frac{1 - \sin^4 r}{1 + \sin^4 r}.$$

As  $r \rightarrow 0$ , the Poincaré duality angle  $\theta_r \rightarrow 0$ .

As  $r \rightarrow \pi/2$ ,  $\theta_r \rightarrow \pi/2$ .

# The Poincaré duality angle for $M_r$

$$\cos \theta_r = \frac{1 - \sin^4 r}{1 + \sin^4 r}.$$

As  $r \rightarrow 0$ , the Poincaré duality angle  $\theta_r \rightarrow 0$ .

As  $r \rightarrow \pi/2$ ,  $\theta_r \rightarrow \pi/2$ .

Generalizes to  $\mathbb{C}P^n - B_r(p)$ .

# Poincaré duality angles of Grassmannians

Consider

$$N_r := G_2\mathbb{R}^n - \nu_r(G_1\mathbb{R}^{n-1}).$$

# Poincaré duality angles of Grassmannians

Consider

$$N_r := G_2\mathbb{R}^n - \nu_r(G_1\mathbb{R}^{n-1}).$$

## Theorem

- As  $r \rightarrow 0$ , all the Poincaré duality angles of  $N_r$  go to zero.
- As  $r$  approaches its maximum value of  $\pi/2$ , all the Poincaré duality angles of  $N_r$  go to  $\pi/2$ .

## Conjecture

*If  $M^n$  is a closed Riemannian manifold and  $N^k$  is a closed submanifold of codimension  $\geq 2$ , the Poincaré duality angles of*

$$M - \nu_r(N)$$

*go to zero as  $r \rightarrow 0$ .*

What can you learn about the topology of  $M$  from knowledge of  $\partial M$ ?

# Electrical Impedance Tomography

Induce potentials on the boundary of a region and determine the conductivity inside the region by measuring the current flux through the boundary.

# Electrical Impedance Tomography

Induce potentials on the boundary of a region and determine the conductivity inside the region by measuring the current flux through the boundary.



# The Voltage-to-Current map

Suppose  $f$  is a potential on the boundary of a region  $M \subset \mathbb{R}^3$ .

# The Voltage-to-Current map

Suppose  $f$  is a potential on the boundary of a region  $M \subset \mathbb{R}^3$ .

Then  $f$  extends to a potential  $u$  on  $M$ , where

$$\Delta u = 0, \quad u|_{\partial M} = f.$$

# The Voltage-to-Current map

Suppose  $f$  is a potential on the boundary of a region  $M \subset \mathbb{R}^3$ .

Then  $f$  extends to a potential  $u$  on  $M$ , where

$$\Delta u = 0, \quad u|_{\partial M} = f.$$

If  $\gamma$  is the conductivity on  $M$ , the current flux through  $\partial M$  is given by

$$(\gamma \nabla u) \cdot \nu = -\gamma \frac{\partial u}{\partial \nu}$$

# The Dirichlet-to-Neumann map

The map  $\Lambda_{\text{cl}} : C^\infty(\partial M) \rightarrow C^\infty(\partial M)$  defined by

$$f \mapsto \frac{\partial u}{\partial \nu}$$

is the classical *Dirichlet-to-Neumann map*.

# The Dirichlet-to-Neumann map

The map  $\Lambda_{\text{cl}} : C^\infty(\partial M) \rightarrow C^\infty(\partial M)$  defined by

$$f \mapsto \frac{\partial u}{\partial \nu}$$

is the classical *Dirichlet-to-Neumann map*.

## Theorem (Lee-Uhlmann)

*If  $M^n$  is a compact, analytic Riemannian manifold with boundary, then  $M$  is determined up to isometry by  $\Lambda_{\text{cl}}$ .*

Joshi–Lionheart and Belishev–Sharafutdinov generalized the classical Dirichlet-to-Neumann map to differential forms:

$$\Lambda : \Omega^p(\partial M) \rightarrow \Omega^{n-p-1}(\partial M)$$

Joshi–Lionheart and Belishev–Sharafutdinov generalized the classical Dirichlet-to-Neumann map to differential forms:

$$\Lambda : \Omega^p(\partial M) \rightarrow \Omega^{n-p-1}(\partial M)$$

If  $\varphi \in \Omega^p(\partial M)$ , then let  $\omega$  solve the boundary value problem

$$\Delta\omega = 0, \quad i^*\omega = \varphi, \quad i^*\delta\omega = 0.$$

# Generalization to differential forms

Joshi–Lionheart and Belishev–Sharafutdinov generalized the classical Dirichlet-to-Neumann map to differential forms:

$$\Lambda : \Omega^p(\partial M) \rightarrow \Omega^{n-p-1}(\partial M)$$

If  $\varphi \in \Omega^p(\partial M)$ , then let  $\omega$  solve the boundary value problem

$$\Delta\omega = 0, \quad i^*\omega = \varphi, \quad i^*\delta\omega = 0.$$

Define

$$\Lambda\varphi := i^* \star d\omega.$$

# Generalization to differential forms

Joshi–Lionheart and Belishev–Sharafutdinov generalized the classical Dirichlet-to-Neumann map to differential forms:

$$\Lambda : \Omega^p(\partial M) \rightarrow \Omega^{n-p-1}(\partial M)$$

If  $\varphi \in \Omega^p(\partial M)$ , then let  $\omega$  solve the boundary value problem

$$\Delta\omega = 0, \quad i^*\omega = \varphi, \quad i^*\delta\omega = 0.$$

Define

$$\Lambda\varphi := i^* \star d\omega.$$

If  $f \in \Omega^0(\partial M)$ ,

$$\Lambda f = i^* \star du = \frac{\partial u}{\partial \nu} \text{dvol}_{\partial M} = (\Lambda_{\text{cl}} f) \text{dvol}_{\partial M}$$

## Theorem (Belishev–Sharafutdinov)

*The data  $(\partial M, \Lambda)$  completely determines the cohomology groups of  $M$ .*

# Connection to Poincaré duality angles

Define the *Hilbert transform*  $T := d\Lambda^{-1}$ .

# Connection to Poincaré duality angles

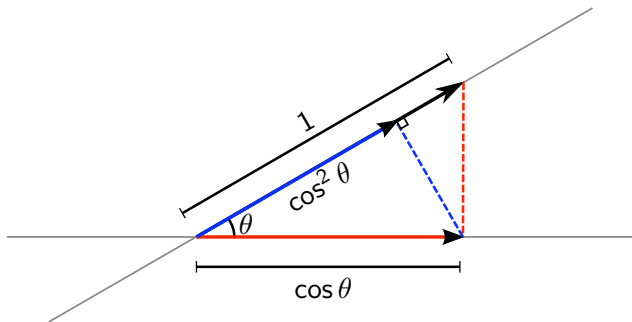
Define the *Hilbert transform*  $T := d\Lambda^{-1}$ .

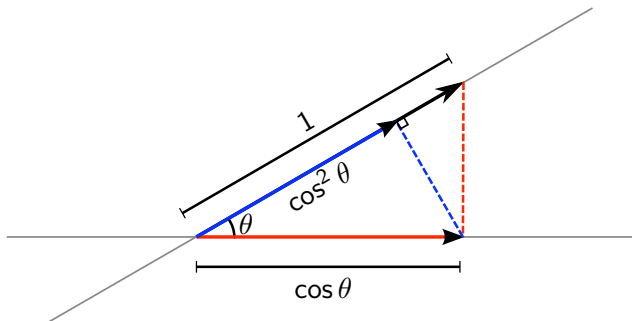
## Theorem

If  $\theta_1, \dots, \theta_k$  are the Poincaré duality angles of  $M$  in dimension  $p$ , then the quantities

$$(-1)^{np+n+p} \cos^2 \theta_i$$

are the non-zero eigenvalues of an appropriate restriction of  $T^2$ .





The Hilbert transform  $T$  recaptures the orthogonal projection  $\mathcal{H}_N^p(M) \rightarrow \mathcal{H}_D^p(M)$ .

Belishev and Sharafutdinov posed the following question:

*Can the multiplicative structure of cohomologies be recovered from our data  $(\partial M, \Lambda)$ ? Till now, the authors cannot answer the question.*

Belishev and Sharafutdinov posed the following question:

*Can the multiplicative structure of cohomologies be recovered from our data  $(\partial M, \Lambda)$ ? Till now, the authors cannot answer the question.*

## Theorem

*The mixed cup product*

$$\cup : H^p(M; \mathbb{R}) \times H^q(M, \partial M; \mathbb{R}) \rightarrow H^{p+q}(M, \partial M; \mathbb{R})$$

*is completely determined by the data  $(\partial M, \Lambda)$  when the relative class is restricted to come from the boundary subspace.*

Thanks!