Higher-dimensional linking integrals

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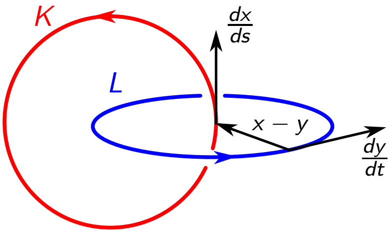
Theorem (Gauß 1833)

Let $K = \{x(s)\}$ and $L = \{y(t)\}$ be disjoint, smooth, closed curves in $\mathbb{R}^3$. Then the linking number between K and L is given by

$$Lk(K, L) = \frac{1}{4\pi} \int_{K \times L} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \frac{x - y}{|x - y|^3} \, ds \, dt.$$



Two ways to look at the Gauß integral

$$\frac{1}{4\pi}\int_{K\times L}\frac{dx}{ds}\times\frac{dy}{dt}\cdot\frac{x-y}{|x-y|^3}\,ds\,dt$$

1. A topological gadget: it computes the linking number.
2. A geometric gadget: the integrand is invariant under orientation-preserving isometries of $\mathbb{R}^3$.

The goal today is to find explicit, computable generalizations of the Gauß linking integral. In particular, such generalized Gauß integrals should be both topological and geometric in the above sense.

A linking integral for spheres

Theorem (Kuperberg, DeTurck–Gluck, S–Vela-Vick)

Let K^k, L^ℓ *be disjoint, oriented, closed submanifolds of* S^n *with* $k + \ell = n - 1$. *Then*

$$Lk(K, L) = \frac{1}{Vol\ S^n} \int_{K \times L} \frac{\Omega_{k,\ell}(\alpha)}{\sin^n \alpha} [x, dx, y, dy]$$

where

$$\Omega_{k,\ell}(\alpha) = \int_{\theta=\alpha}^{\pi} \sin^k(\theta - \alpha) \sin^\ell \theta d\theta$$

and $\alpha(x, y)$ *is the geodesic distance in* S^n *from* $x \in K$ *to* $y \in L$.

Remark

The integrand is invariant under orientation-preserving isometries of S^n.

The linking integral for spheres comes as a corollary of a more general theorem. Before stating it, we need some terminology.

Definition

A smooth hypersurface $M^n \subset \mathbb{R}^{n+1}$ is called a *visible hypersurface* if every ray from the origin either misses M completely or hits it exactly once, transversally.

Examples

- $S^n \subset \mathbb{R}^{n+1}$
- $S^n \times \mathbb{R}^m \subset \mathbb{R}^{n+1} \times \mathbb{R}^m = \mathbb{R}^{n+m+1}$ (in particular, $S^2 \times \mathbb{R}$)
- All closed manifolds are homotopy equivalent to a complete visible hypersurface in some Euclidean space

Theorem (S–Vela-Vick)

Let K^k, L^ℓ be disjoint, closed, oriented, nullhomologous submanifolds of a visible hypersurface M^n with $k+\ell = n-1$. Then

$$Lk(K,L) = \frac{1}{Vol\ S^n} \int_{K\times L} \frac{\Omega_{k,\ell}(\alpha)}{|x|^k|y|^\ell \sin^n \alpha}\,[x, dx, y, dy],$$

where $\Omega_{k,\ell}(\alpha)$ is as before and $\alpha(x,y)$ is the Euclidean angle between $x \in K$ and $y \in L$.

Remark

This integrand is $SO(n+1)$-invariant: for $h \in SO(n+1)$, the integrand is the same for $h(K)$ and $h(L)$ as for K and L, even if $h(M) \neq M$.

Proofs of the Gauß linking integral

Two proofs of the Gauß linking integral:

Electrodynamics Think of K and L as wires, run a unit current through K, and use Ampère's Law.
This proof can be generalized to other 3-manifolds (DeTurck–Gluck: S^3 and H^3), but not higher dimensions.

Degree of map Let $f : K \times L \longrightarrow S^2$ be given by $(s,t) \mapsto \frac{x(s)-y(t)}{|x(s)-y(t)|}$.
Then

$$Lk(K,L) = -\deg(f) = -\frac{1}{\text{Vol } S^2} \int_{K \times L} f^* \omega,$$

where ω is the $SO(3)$-invariant volume form on S^2.

The degree-of-map proof of the Gauß linking integral easily generalizes to higher-dimensional Euclidean spaces:

Proposition

Let K^k, L^ℓ be disjoint, closed, oriented submanifolds of $\mathbb{R}^N$ such that $k + \ell = N - 1$. Let $x(\mathbf{s}) : \mathbb{R}^k \to K$ and $y(\mathbf{t}) : \mathbb{R}^\ell \to L$ be oriented local coordinates for K and L. If $f : K \times L \to S^{N-1}$ is given by

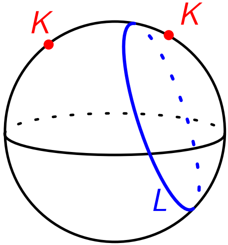
$$(\mathbf{s}, \mathbf{t}) \mapsto \frac{x(\mathbf{s}) - y(\mathbf{t})}{|x(\mathbf{s}) - y(\mathbf{t})|}$$

Then

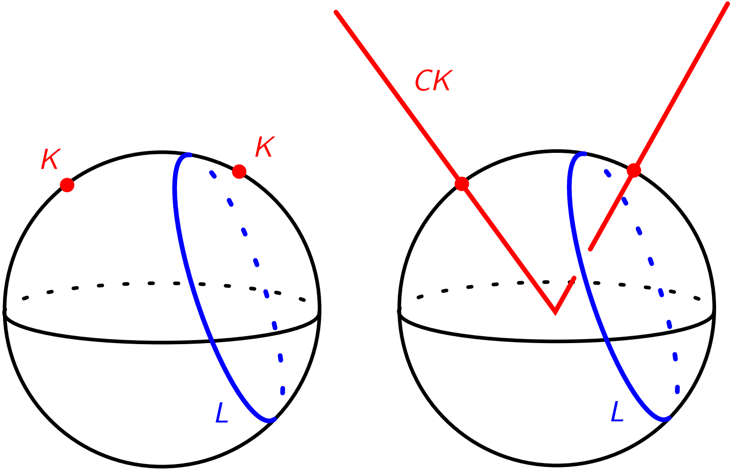
$$Lk(K, L) = (-1)^N \deg(f) = \frac{(-1)^{k+1}}{\text{Vol } S^{N-1}} \int_{K \times L} \frac{1}{|x - y|^N} \, [x - y, dx, dy].$$

Use what we know (the linking integral in Euclidean space) to get what we want (a linking integral in spheres or, more generally, visible hypersurfaces).

We can't just use the Euclidean integral directly; submanifolds of a hypersurface have the wrong codimension to be linked in the ambient Euclidean space.

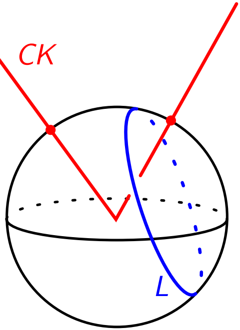


The idea is to bump up the dimension of one of the submanifolds, use the Euclidean integral, then integrate out the extra dimension and hope for something intrinsic.


K
K
L
CK
L

We bump up the dimension of K by taking the half-infinite cone from the origin:

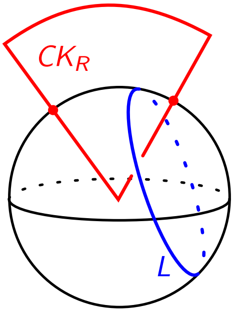
$$CK := \{\tau x \mid x \in K, \tau \in [0, \infty)\}$$



Then $Lk_{M^n}(K, L) = Lk_{\mathbb{R}^{n+1}}(CK, L)$, which we can compute using the linking integral in $\mathbb{R}^{n+1}$.

1. CK is not compact
   Let $K_R := \{\tau x \mid x \in K, \tau \in [0, R]\}$ and let $\overline{K}$ be a chain in M bounded by K. Define

$$CK_R := K_R \cup R\overline{K}$$



2. CK_R is not a manifold
   We are integrating, so singularities of measure zero can be dealt with.

$Lk_{\mathbb{R}^{n+1}}(CK_R, L)$ is now well-defined, and, by construction,

$$\begin{aligned} Lk_{M^n}(K, L) &= Lk_{\mathbb{R}^{n+1}}(CK_R, L) \\ &= \frac{(-1)^{(k+1)+1}}{\mathrm{Vol}\ S^n} \int_{K_R \times L} \frac{1}{|\tau x - y|^{n+1}} [\tau x - y, d(\tau x), dy] \\ &\quad + \frac{(-1)^{(k+1)+1}}{\mathrm{Vol}\ S^n} \int_{\overline{K} \times L} \frac{1}{|Rz - y|^{n+1}} [Rz - y, d(Rz), dy]. \end{aligned}$$

As $R \to \infty$, the second integral goes to zero since there are $k+2$ R's in the numerator and $n+1$ in the denominator.
As $R \to \infty$, the first integral goes to

$$\frac{(-1)^k}{\mathrm{Vol}\ S^n} \int_{CK \times L} \frac{1}{|\tau x - y|^{n+1}} [\tau x - y, d(\tau x), dy].$$

Splitting the integrand

We have

$$Lk(K,L) = \frac{(-1)^k}{\text{Vol } S^n} \int_{CK\times L} \frac{1}{|\tau x - y|^{n+1}} [\tau x - y, d(\tau x), dy].$$

The form splits as

$$\begin{aligned} [\tau x - y, d(\tau x), dy] &= d\tau \wedge [\tau x - y, x, \tau dx, dy] \\ &= (-1)^k \tau^k d\tau \wedge [x, dx, y, dy], \end{aligned}$$

so we can re-write $Lk(K,L)$ as

$$Lk(K,L) = \frac{1}{\text{Vol } S^n} \int_{K\times L} \left(\int_{\tau=0}^{\infty} \frac{\tau^k}{|\tau x - y|^{n+1}} \, d\tau \right) [x, dx, y, dy]$$

We have

$$Lk(K, L) = \frac{1}{\text{Vol } S^n} \int_{K\times L} \left(\int_{\tau=0}^{\infty} \frac{\tau^k}{|\tau x - y|^{n+1}} \, d\tau \right) [x, dx, y, dy]$$

and we want to show that

$$Lk(K, L) = \frac{1}{\text{Vol } S^n} \int_{K\times L} \frac{\Omega_{k,\ell}(\alpha)}{|x|^k |y|^\ell \sin^n \alpha} [x, dx, y, dy],$$

Hence, we need only show that

$$\int_{\tau=0}^{\infty} \frac{\tau^k}{|\tau x - y|^{n+1}} d\tau = \frac{1}{|x|^k |y|^\ell \sin^n \alpha} \int_{\theta=\alpha}^{\pi} \sin^k(\theta - \alpha) \sin^\ell \theta d\theta.$$

Freshman Calculus!

Thanks