

Higher-dimensional linking integrals

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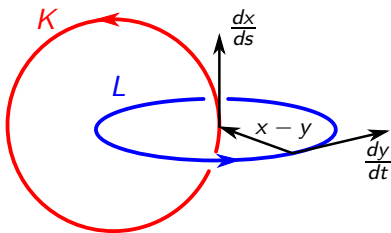
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The Gauß Linking Integral

Theorem (Gauß 1833)

Let $K = \{x(s)\}$ and $L = \{y(t)\}$ be disjoint, smooth, closed curves in \mathbb{R}^3 . Then the linking number between K and L is given by

$$Lk(K, L) = \frac{1}{4\pi} \int_{K \times L} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \frac{x - y}{|x - y|^3} ds dt.$$



Two ways to look at the Gauß integral

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- 2 A geometric gadget: the integrand is invariant under orientation-preserving isometries of \mathbb{R}^3 .

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The goal today is to find explicit, computable generalizations of the Gauß linking integral. In particular, such generalized Gauß integrals should be both topological and geometric in the above sense.

Theorem (Kuperberg, DeTurck–Gluck, S–Vela–Vick)

Let K^k, L^ℓ be disjoint, oriented, closed submanifolds of S^n with $k + \ell = n - 1$. Then

$$Lk(K, L) = \frac{1}{\text{Vol } S^n} \int_{K \times L} \frac{\Omega_{k,\ell}(\alpha)}{\sin^n \alpha} [x, dx, y, dy]$$

where

$$\Omega_{k,\ell}(\alpha) = \int_{\theta=\alpha}^{\pi} \sin^k(\theta - \alpha) \sin^\ell \theta d\theta$$

and $\alpha(x, y)$ is the geodesic distance in S^n from $x \in K$ to $y \in L$.

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Remark

The integrand is invariant under orientation-preserving isometries of S^n .

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- All closed manifolds are homotopy equivalent to a complete visible hypersurface in some Euclidean space

Theorem (S-Vela-Vick)

Let K^k, L^ℓ be disjoint, closed, oriented, nullhomologous submanifolds of a visible hypersurface M^n with $k + \ell = n - 1$. Then

$$Lk(K, L) = \frac{1}{\text{Vol } S^n} \int_{K \times L} \frac{\Omega_{k,\ell}(\alpha)}{|x|^k |y|^\ell \sin^n \alpha} [x, dx, y, dy],$$

where $\Omega_{k,\ell}(\alpha)$ is as before and $\alpha(x, y)$ is the Euclidean angle between $x \in K$ and $y \in L$.

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Remark

This integrand is $SO(n+1)$ -invariant: for $h \in SO(n+1)$, the integrand is the same for $h(K)$ and $h(L)$ as for K and L , even if $h(M) \neq M$.

Proofs of the Gauß linking integral

Two proofs of the Gauß linking integral:

Electrodynamics Think of K and L as wires, run a unit current through K , and use Ampère's Law.

This proof can be generalized to other 3-manifolds (DeTurck–Gluck: S^3 and H^3), but not higher dimensions.

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Degree of map Let $f : K \times L \rightarrow S^2$ be given by $(s, t) \mapsto \frac{x(s)-y(t)}{|x(s)-y(t)|}$. Then

$$Lk(K, L) = -\deg(f) = -\frac{1}{\text{Vol } S^2} \int_{K \times L} f^* \omega,$$

where ω is the $SO(3)$ -invariant volume form on S^2 .

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Proposition

Let K^k, L^ℓ be disjoint, closed, oriented submanifolds of \mathbb{R}^N such that $k + \ell = N - 1$. Let $x(\mathbf{s}) : \mathbb{R}^k \rightarrow K$ and $y(\mathbf{t}) : \mathbb{R}^\ell \rightarrow L$ be oriented local coordinates for K and L . If $f : K \times L \rightarrow S^{N-1}$ is given by

$$(\mathbf{s}, \mathbf{t}) \mapsto \frac{x(\mathbf{s}) - y(\mathbf{t})}{|x(\mathbf{s}) - y(\mathbf{t})|}$$

Then

$$Lk(K, L) = (-1)^N \deg(f) = \frac{(-1)^{k+1}}{\text{Vol } S^{N-1}} \int_{K \times L} \frac{1}{|x - y|^N} [x - y, dx, dy].$$

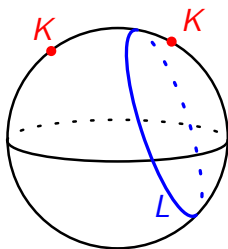
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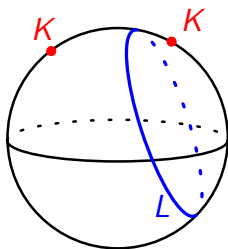
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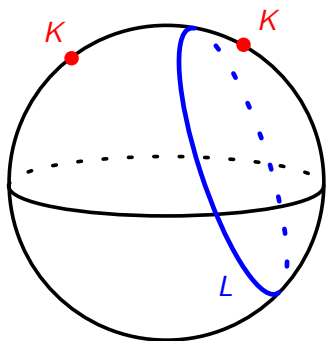


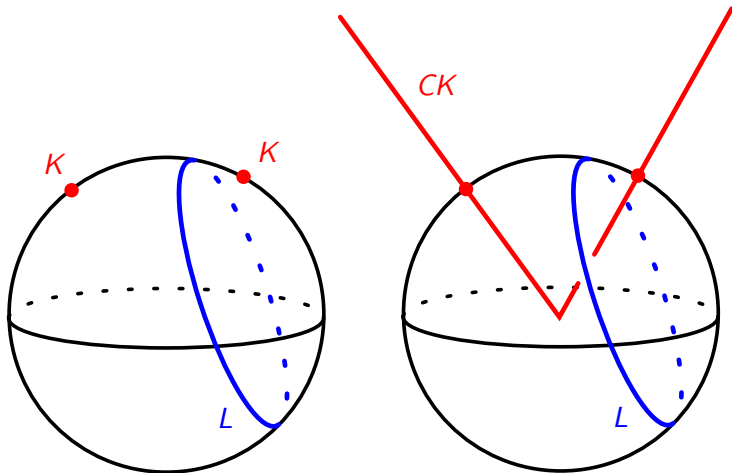
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The idea is to bump up the dimension of one of the submanifolds, use the Euclidean integral, then integrate out the extra dimension and hope for something intrinsic.

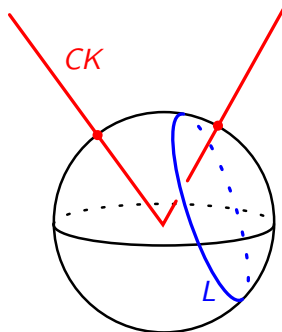




Bumping up the dimension of K

We bump up the dimension of K by taking the half-infinite cone from the origin:

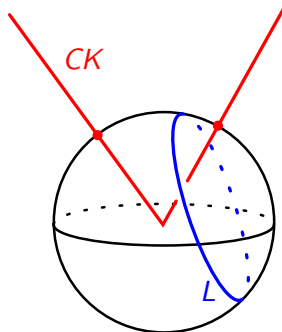
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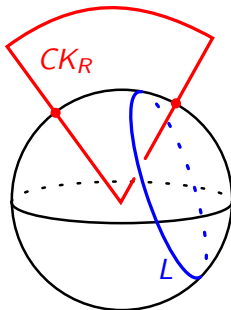
Then $Lk_{M^n}(K, L) = Lk_{\mathbb{R}^{n+1}}(CK, L)$, which we can compute using the linking integral in \mathbb{R}^{n+1} .

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Let $K_R := \{\tau x \mid x \in K, \tau \in [0, R]\}$ and let \overline{K} be a chain in M bounded by K . Define

$$CK_R := K_R \cup R\overline{K}$$

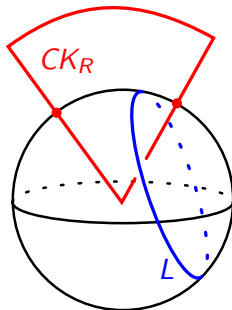


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We are integrating, so singularities of measure zero can be dealt with.

$Lk_{\mathbb{R}^{n+1}}(CK_R, L)$ is now well-defined, and, by construction,

$$\begin{aligned} Lk_{M^n}(K, L) &= Lk_{\mathbb{R}^{n+1}}(CK_R, L) \\ &= \frac{(-1)^{(k+1)+1}}{\text{Vol } S^n} \int_{K_R \times L} \frac{1}{|\tau x - y|^{n+1}} [\tau x - y, d(\tau x), dy] \\ &\quad + \frac{(-1)^{(k+1)+1}}{\text{Vol } S^n} \int_{\bar{K} \times L} \frac{1}{|Rz - y|^{n+1}} [Rz - y, d(Rz), dy]. \end{aligned}$$

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As $R \rightarrow \infty$, the second integral goes to zero since there are $k + 2$ R 's in the numerator and $n + 1$ in the denominator.

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As $R \rightarrow \infty$, the first integral goes to

$$\frac{(-1)^k}{\text{Vol } S^n} \int_{CK \times L} \frac{1}{|\tau x - y|^{n+1}} [\tau x - y, d(\tau x), dy].$$

We have

$$Lk(K, L) = \frac{(-1)^k}{\text{Vol } S^n} \int_{CK \times L} \frac{1}{|\tau x - y|^{n+1}} [\tau x - y, d(\tau x), dy].$$

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The form splits as

$$\begin{aligned} [\tau x - y, d(\tau x), dy] &= d\tau \wedge [\tau x - y, x, \tau dx, dy] \\ &= (-1)^k \tau^k d\tau \wedge [x, dx, y, dy], \end{aligned}$$

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so we can re-write $Lk(K, L)$ as

$$Lk(K, L) = \frac{1}{\text{Vol } S^n} \int_{K \times L} \left(\int_{\tau=0}^{\infty} \frac{\tau^k}{|\tau x - y|^{n+1}} d\tau \right) [x, dx, y, dy]$$

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Freshman Calculus!

Thanks