

# Poincaré Duality Angles on Riemannian Manifolds with Boundary

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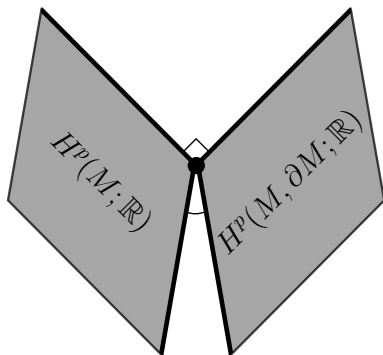
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## de Rham's Theorem

*Suppose  $M^n$  is a compact, oriented, smooth manifold. Then*

$$H^p(M; \mathbb{R}) \cong \mathcal{C}^p(M) / \mathcal{E}^p(M),$$

*where  $\mathcal{C}^p(M)$  is the space of closed  $p$ -forms on  $M$  and  $\mathcal{E}^p(M)$  is the space of exact  $p$ -forms.*

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Kodaira called this the space of *harmonic  $p$ -fields* on  $M$ .

## Hodge's Theorem

*If  $M^n$  is a closed, oriented, smooth Riemannian manifold,*

$$H^p(M; \mathbb{R}) \cong \mathcal{H}^p(M).$$

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Then

$$H^p(M; \mathbb{R}) \cong \mathcal{H}_N^p(M).$$

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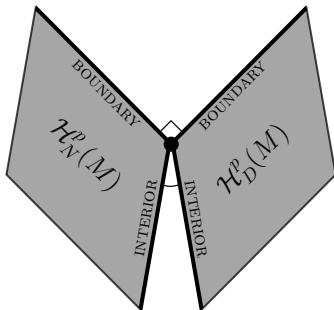
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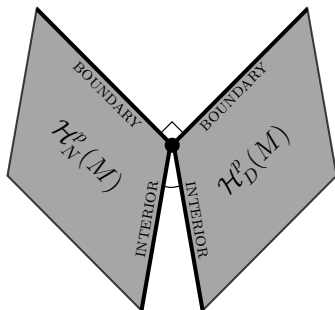
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## Definition (DeTurck–Gluck)

The *Poincaré duality angles* of the Riemannian manifold  $M$  are the principal angles between the interior subspaces.

# What do the Poincaré duality angles tell you?

## Guess

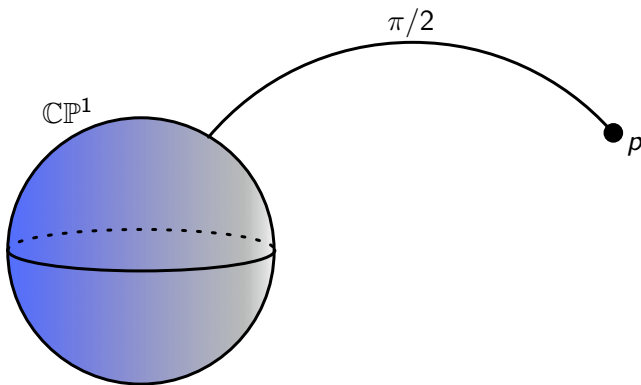
*If  $M$  is “almost” closed, the Poincaré duality angles of  $M$  should be small.*

Consider  $\mathbb{C}\mathbb{P}^2$  with its usual Fubini-Study metric. Let  $p \in \mathbb{C}\mathbb{P}^2$ .  
Then define

$$M_r := \mathbb{C}\mathbb{P}^2 - B_r(p).$$

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$M_r$  has relative cohomology in dimensions 2 and 4.

Therefore,  $M_r$  has a single Poincaré duality angle  $\theta_r$  between  $\mathcal{H}_N^2(M_r)$  and  $\mathcal{H}_D^2(M_r)$ .

## Find harmonic 2-fields

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Find closed and co-closed  $SU(2)$ -invariant forms on  $M_r$  satisfying Neumann and Dirichlet boundary conditions

# The Poincaré duality angle for $M_r$

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Generalizes to  $\mathbb{C}P^n - B_r(p)$ .

# Poincaré duality angles of Grassmannians

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## Theorem

- As  $r \rightarrow 0$ , all the Poincaré duality angles of  $N_r$  go to zero.
- As  $r$  approaches its maximum value of  $\pi/2$ , all the Poincaré duality angles of  $N_r$  go to  $\pi/2$ .

## Conjecture

*If  $M^n$  is a closed Riemannian manifold and  $N^k$  is a closed submanifold of codimension  $\geq 2$ , the Poincaré duality angles of*

$$M - \nu_r(N)$$

*go to zero as  $r \rightarrow 0$ .*

What can you learn about the topology of  $M$  from knowledge of  $\partial M$ ?

# Electrical Impedance Tomography

Induce potentials on the boundary of a region and determine the conductivity inside the region by measuring the current flux through the boundary.

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If  $\gamma$  is the conductivity on  $M$ , the current flux through  $\partial M$  is given by

$$(\gamma \nabla u) \cdot \nu = -\gamma \frac{\partial u}{\partial \nu}$$

# The Dirichlet-to-Neumann map

The map  $\Lambda_{\text{cl}} : C^\infty(\partial M) \rightarrow C^\infty(\partial M)$  defined by

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## Theorem (Lee-Uhlmann)

*If  $M^n$  is a compact, analytic Riemannian manifold with boundary, then  $M$  is determined up to isometry by  $\Lambda_{\text{cl}}$ .*

# Generalization to differential forms

Joshi–Lionheart and Belishev–Sharafutdinov generalized the classical Dirichlet-to-Neumann map to differential forms:

$$\Lambda : \Omega^p(\partial M) \rightarrow \Omega^{n-p-1}(\partial M)$$

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## Theorem (Belishev–Sharafutdinov)

*The data  $(\partial M, \Lambda)$  completely determines the cohomology groups of  $M$ .*

# Connection to Poincaré duality angles

Define the *Hilbert transform*  $T := d\Lambda^{-1}$ .

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## Theorem

If  $\theta_1, \dots, \theta_k$  are the Poincaré duality angles of  $M$  in dimension  $p$ , then the quantities

$$(-1)^{np+n+p} \cos^2 \theta_i$$

are the non-zero eigenvalues of an appropriate restriction of  $T^2$ .

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## Theorem

*The mixed cup product*

$$\cup : H^p(M; \mathbb{R}) \times H^q(M, \partial M; \mathbb{R}) \rightarrow H^{p+q}(M, \partial M; \mathbb{R})$$

*is completely determined by the data  $(\partial M, \Lambda)$  when the relative class is restricted to come from the boundary subspace.*

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What about other cup products?
- Can the  $L^2$  inner product on  $\mathcal{H}_N^p(M)$  and  $\mathcal{H}_D^p(M)$  be recovered from  $(\partial M, \Lambda)$ ?

Thanks!