

# THE COMPLETE DIRICHLET-TO-NEUMANN MAP FOR DIFFERENTIAL FORMS

VLADIMIR SHARAFUTDINOV AND CLAYTON SHONKWILER

ABSTRACT. The Dirichlet-to-Neumann map for differential forms on a Riemannian manifold with boundary is a generalization of the classical Dirichlet-to-Neumann map which arises in the problem of Electrical Impedance Tomography. We synthesize the two different approaches to defining this operator by giving an invariant definition of the complete Dirichlet-to-Neumann map for differential forms in terms of two linear operators  $\Phi$  and  $\Psi$ . The pair  $(\Phi, \Psi)$  is equivalent to Joshi and Lionheart's operator  $\Pi$  and determines Belishev and Sharafutdinov's operator  $\Lambda$ . We show that the Betti numbers of the manifold are determined by  $\Phi$  and that  $\Psi$  determines a chain complex whose homologies are explicitly related to the cohomology groups of the manifold.

## 1. INTRODUCTION

We consider the problem of recovering the topology of a compact, oriented, smooth Riemannian manifold  $(M, g)$  with boundary from the Dirichlet-to-Neumann map for differential forms. The classical Dirichlet-to-Neumann map for functions was first defined by Calderón [Cal80], and has been shown to recover surfaces up to conformal equivalence [LU01, Bel03] and real-analytic manifolds of dimension  $\geq 3$  up to isometry [LTU03].

The classical Dirichlet-to-Neumann map was generalized to an operator on differential forms independently by Joshi and Lionheart [JL05] and Belishev and Sharafutdinov [BS08]. Joshi and Lionheart called their operator  $\Pi$  and showed that the data  $(\partial M, \Pi)$  determines the  $C^\infty$ -jet of the Riemannian metric at the boundary. Krupchyk, Lassas, and Uhlmann have recently extended this result to show that  $(\partial M, \Pi)$  determines a real-analytic manifold up to isometry [KLU10].

On the other hand, Belishev and Sharafutdinov called their Dirichlet-to-Neumann map  $\Lambda$  and showed that  $(\partial M, \Lambda)$  determines the cohomology groups of the manifold  $M$ . Shonkwiler [Sho09] demonstrated a connection between  $\Lambda$  and invariants called Poincaré duality angles and showed that the cup product structure of the manifold  $M$  can be partially recovered from  $(\partial M, \Lambda)$ .

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The operators  $\Pi$  and  $\Lambda$  are similar, but do not appear to be equivalent. One of the advantages of Belishev and Sharafutdinov's  $\Lambda$ , especially for the task of recovering topological data, is that it is defined invariantly. In this paper we provide an invariant definition of Joshi and Lionheart's operator  $\Pi$ , which we give in terms of two auxiliary operators

$$\Phi : \Omega^k(\partial M) \rightarrow \Omega^{n-k-1}(\partial M) \quad \text{and} \quad \Psi : \Omega^k(\partial M) \rightarrow \Omega^{k-1}(\partial M).$$

We can easily show that  $\Lambda$  is determined by  $\Phi$  and  $\Psi$ , so it makes sense to regard  $\Pi$  as the “complete” Dirichlet-to-Neumann operator on differential forms.

Belishev and Sharafutdinov's proof that the Betti numbers of  $M$  can be recovered from the data  $(\partial M, \Lambda)$  was somewhat circuitous, as it involved determining the dimension of the image of the operator  $G = \Lambda \pm d_\partial \Lambda^{-1} d_\partial$ . In contrast, it is straightforward to recover the Betti numbers of  $M$  from  $\Phi$ .

**Theorem 1.** *Let  $\beta_k(M) = \dim H^k(M; \mathbb{R})$  be the  $k$ th Betti number of  $M$ . Then*

$$\beta_k(M) = \dim \ker \Phi.$$

The operator  $\Psi$  turns out to be a chain map and the homology of the chain complex  $(\Omega^*(\partial M), \Psi)$  is given in terms of a mixture of absolute and relative cohomology groups of  $M$ .

**Theorem 2.** *For any  $0 \leq k \leq n-1$ ,*

$$H_k(\Omega^*(\partial M), \Psi) \simeq H^{k+1}(M, \partial M; \mathbb{R}) \oplus H^k(M; \mathbb{R}).$$

This, in turn, implies that the space of  $k$ -forms on  $\partial M$  contains an “echo” (detected by  $\Pi$ ) of the  $(k+1)$ st relative cohomology group of  $M$ .

**Corollary 3.** *The space  $\Omega^k(\partial M)$  of  $k$ -forms on  $\partial M$  contains a subspace isomorphic to  $H^{k+1}(M, \partial M; \mathbb{R})$  which is distinguished by the Dirichlet-to-Neumann operator  $\Pi$ . Specifically,*

$$(\ker \Psi_k / \text{im } \Psi_{k+1}) / \ker \Phi_k \simeq H^{k+1}(M, \partial M; \mathbb{R}).$$

When  $n = 2$  and  $k = 0$ , Theorem 1 and Corollary 3 imply that all the cohomology groups of a surface are contained in  $\Omega^0(\partial M)$ .

**Corollary 4.** *All of the cohomology groups of a surface  $M$  with boundary can be realized inside the space of smooth functions on  $\partial M$ , where they can be recovered by the Dirichlet-to-Neumann operator  $\Pi$ .*

Since  $\Psi$  is a chain map, it is natural to try to define associated cochain maps and compute their cohomologies. In this spirit, we define  $\tilde{\Psi} = \pm \star_\partial \Psi \star_\partial$  and show that it is the adjoint of  $\Psi$ . Not surprisingly,

$$H^k(\Omega^*(\partial M), \tilde{\Psi}) \simeq H_{n-k-1}(\Omega^*(\partial M), \Psi).$$

Finally, we define another cochain map  $\Theta$  with the same cohomology as  $\tilde{\Psi}$ . It turns out that  $\Theta = \pm d_\partial \Phi^2$ , so the cohomology of  $\tilde{\Psi}$  (and hence the homology of  $\Psi$ ) is completely determined by the operator  $\Phi$ . With this in

mind, restating Corollary 3 in terms of  $\Phi$  and specializing to the case  $k = 0$  yields the following:

**Corollary 5.** *A copy of the cohomology group  $H^{n-1}(M; \mathbb{R})$  is distinguished by the operator  $\Phi$  inside  $\Omega^0(\partial M)$ , the space of smooth functions on  $\partial M$ . Specifically,*

$$\ker(d_{\partial}\Phi^2)/\ker\Phi \simeq H^{n-1}(M; \mathbb{R}).$$

The above results all suggest that the operator  $\Pi$  (and, in particular,  $\Phi$ ) encodes more information about the topology of  $M$  than does the operator  $\Lambda$ . Thus far nobody has been able to use  $\Lambda$  to recover the cohomology ring structure on  $M$ , but perhaps this will be easier to recover from the operator  $\Pi$ . Another interesting question relates to the linearized inverse problem of recovering the metric: can the results of [Sha09] be strengthened if the data  $\Lambda$  are replaced with the richer data  $(\Phi, \Psi)$ ?

## 2. THE OPERATORS $\Phi$ AND $\Psi$

Throughout this paper,  $(M, g)$  will be a smooth, compact, oriented Riemannian manifold of dimension  $n \geq 2$  with nonempty boundary. The term “smooth” is used as a synonym for “ $C^\infty$ -smooth”. Let  $i : \partial M \hookrightarrow M$  be the identical embedding and let  $\Omega(M) = \bigoplus_{k=0}^n \Omega^k(M)$  be the graded algebra of smooth differential forms on  $M$ . We use the standard operators  $d, \delta, \Delta$ , and  $\star$  on  $\Omega(M)$ , as well as their analogues  $d_{\partial}, \delta_{\partial}, \Delta_{\partial}$ , and  $\star_{\partial}$  on  $\Omega(\partial M)$ .

Joshi and Lionheart defined their Dirichlet-to-Neumann operator

$$\Pi : \Omega(M)|_{\partial M} \rightarrow \Omega(M)|_{\partial M}$$

as

$$\Pi\chi := \left. \frac{\partial\omega}{\partial\nu} \right|_{\partial M},$$

where  $\nu$  is the unit outward normal vector at the boundary and  $\omega$  is the solution to the boundary value problem

$$\begin{cases} \Delta\omega = 0 \\ \omega|_{\partial M} = \chi. \end{cases}$$

This boundary value problem has a unique solution for every  $\chi \in \Omega(M)|_{\partial M}$  [Sch95, Theorem 3.4.1].

When applied to forms, the meaning of the normal derivative  $\partial/\partial\nu$  needs to be specified. Instead, we prefer to give an equivalent definition of  $\Pi$  in invariant terms. To do so, note that the restriction  $\omega|_{\partial M}$  is determined by two boundary forms,  $i^*\omega$  and  $i^*\star\omega$ . Likewise, the data  $\partial\omega/\partial\nu|_{\partial M}$  are equivalent to the two boundary forms  $i^*\star d\omega$  and  $i^*\delta\omega$ . Hence, we will define the operator

$$\Pi : \Omega^k(\partial M) \times \Omega^{n-k}(\partial M) \rightarrow \Omega^{n-k-1}(\partial M) \times \Omega^{k-1}(\partial M)$$

by

$$(1) \quad \Pi \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} i^* \star d\omega \\ i^* \delta\omega \end{pmatrix}$$

where  $\omega \in \Omega^k(M)$  is the solution to the boundary value problem

$$(2) \quad \begin{cases} \Delta\omega = 0 \\ i^* \omega = \varphi, \quad i^* \star \omega = \psi. \end{cases}$$

Since  $\Pi$  sends pairs of forms to pairs of forms, it is somewhat cumbersome to work with in practice. Instead of using it directly, we find a pair of operators  $(\Phi, \Psi)$  which is equivalent to  $\Pi$ . Define the linear operators

$$\Phi : \Omega^k(\partial M) \rightarrow \Omega^{n-k-1}(\partial M) \quad \text{and} \quad \Psi : \Omega^k(\partial M) \rightarrow \Omega^{k-1}(\partial M)$$

by the equalities

$$(3) \quad \Phi\varphi = i^* \star d\omega \quad \text{and} \quad \Psi\varphi = i^* \delta\omega.$$

Here  $\omega \in \Omega^k(M)$  is the solution to the boundary value problem

$$(4) \quad \begin{cases} \Delta\omega = 0 \\ i^* \omega = \varphi, \quad i^* \star \omega = 0. \end{cases}$$

Now it is straightforward to express  $\Pi$  in terms of  $\Phi$  and  $\Psi$ . We write  $\Pi$  as the matrix

$$\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{pmatrix}.$$

Then, comparing (1) and (3),

$$\Pi_{11} = \Phi, \quad \Pi_{21} = \Psi.$$

From (1) and (2), the operators  $\Pi_{12}$  and  $\Pi_{22}$  are given by

$$\Pi_{12}\psi = i^* \star d\varepsilon \quad \text{and} \quad \Pi_{22}\psi = i^* \delta\varepsilon,$$

where  $\varepsilon$  solves the boundary value problem

$$\begin{cases} \Delta\varepsilon = 0 \\ i^* \varepsilon = 0, \quad i^* \star \varepsilon = \psi. \end{cases}$$

If  $\varepsilon \in \Omega^k(M)$  is the solution to this boundary value problem for  $\psi \in \Omega^{n-k}(\partial M)$ , then the form  $\omega = \star\varepsilon$  solves the problem

$$\begin{cases} \Delta\omega = 0 \\ i^* \omega = \psi, \quad i^* \star \omega = 0. \end{cases}$$

Comparing this to (4), we see that

$$(5) \quad \Phi\psi = i^* \star d\omega \quad \text{and} \quad \Psi\psi = i^* \delta\omega.$$

Since

$$i^* \star d\omega = (-1)^{n(n-k)+1} i^* \delta\varepsilon \quad \text{and} \quad i^* \delta\omega = (-1)^{k+1} i^* \star d\varepsilon,$$

(1) and (5) imply that

$$\Pi_{12} = (-1)^{n(n-k)+1}\Psi \quad \text{and} \quad \Pi_{22} = (-1)^{k+1}\Phi \quad \text{on} \quad \Omega^{n-k}(\partial M).$$

Therefore, the operator  $\Pi$  can be expressed in terms of  $\Phi$  and  $\Psi$  as

$$(6) \quad \Pi = \begin{pmatrix} \Phi & (-1)^{n(n-k)+1}\Psi \\ \Psi & (-1)^{k+1}\Phi \end{pmatrix} \quad \text{on} \quad \Omega^k(\partial M) \times \Omega^{n-k}(\partial M).$$

Belishev and Sharafutdinov's version of the Dirichlet-to-Neumann map is the operator

$$\Lambda : \Omega^k(\partial M) \rightarrow \Omega^{n-k-1}(\partial M)$$

given by

$$\Lambda\varphi = i^* \star d\omega,$$

where  $\omega \in \Omega^k(M)$  is a solution to the boundary value problem

$$(7) \quad \begin{cases} \Delta\omega = 0 \\ i^*\omega = \varphi, \quad i^*\delta\omega = 0. \end{cases}$$

We can now express the operator  $\Lambda$  in terms of  $\Phi$  and  $\Psi$ . Given  $\varphi \in \Omega^k(\partial M)$ , let  $\omega \in \Omega^k(M)$  solve the boundary value problem (7) and set  $\psi = i^* \star \omega$ . Then  $\omega$  solves the boundary value problem (2), so we have that

$$\Pi \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} i^* \star d\omega \\ i^*\delta\omega \end{pmatrix} = \begin{pmatrix} \Lambda\varphi \\ 0 \end{pmatrix}.$$

With the help of (6) we can rewrite this equation as the system

$$\begin{aligned} \Phi\varphi + (-1)^{n(n-k)+1}\Psi\psi &= \Lambda\varphi \\ \Psi\varphi + (-1)^{k+1}\Phi\psi &= 0. \end{aligned}$$

Eliminating  $\psi$  from the system yields the expression

$$(8) \quad \Lambda = \Phi + (-1)^{n(n-k)+k+1}\Psi\Phi^{-1}\Psi \quad \text{on} \quad \Omega^k(\partial M).$$

The fact that the operator  $\Psi\Phi^{-1}\Psi$  is well-defined follows from Corollary 4.3, stated below.

We take this opportunity to record some useful relations involving  $\Phi$  and  $\Psi$ :

**Lemma 2.1.** *The operators  $\Phi$  and  $\Psi$  satisfy the following relations:*

$$(9) \quad \Phi\Psi = (-1)^k d_\partial \Phi \quad \text{on} \quad \Omega^k(\partial M),$$

$$(10) \quad \Psi^2 = 0$$

$$(11) \quad \Psi\Phi = (-1)^{k+1}\Phi d_\partial \quad \text{on} \quad \Omega^k(\partial M),$$

$$(12) \quad \Phi^2 = (-1)^{kn}(d_\partial \Psi + \Psi d_\partial) \quad \text{on} \quad \Omega^k(\partial M)$$

*Proof.* Given  $\varphi \in \Omega^k(\partial M)$ , let  $\omega \in \Omega^k(M)$  solve the boundary value problem (4). Then

$$(13) \quad \Phi\varphi = i^*\star\omega, \quad \Psi\varphi = i^*\delta\omega.$$

Letting  $\xi = \delta\omega$ , we certainly have  $\Delta\xi = 0$ . Pulling  $\xi$  and  $\star\xi$  back to the boundary yields

$$\begin{aligned} i^*\xi &= i^*\delta\omega = \Psi\varphi \\ i^*\star\xi &= i^*\star\delta\omega = \pm i^*d\star\omega = \pm d_{\partial}i^*\star\omega = 0. \end{aligned}$$

Therefore,  $\xi$  solves the boundary value problem

$$\begin{cases} \Delta\xi = 0 \\ i^*\xi = \Psi\varphi, \quad i^*\star\xi = 0, \end{cases}$$

and so

$$(14) \quad \Phi\Psi\varphi = i^*\star d\xi \quad \text{and} \quad \Psi^2\varphi = i^*\delta\xi.$$

Since  $\Delta\omega = 0$ , it follows that  $d\delta\omega = -\delta d\omega$ , which we use to see that

$$\begin{aligned} i^*\star d\xi &= i^*\star d\delta\omega = -i^*\star\delta d\omega = (-1)^k i^*d\star d\omega = (-1)^k d_{\partial}i^*\star d\omega, \\ i^*\delta\xi &= i^*\delta\delta\omega = 0. \end{aligned}$$

Comparing this with (14), we obtain

$$\Phi\Psi\varphi = (-1)^k d_{\partial}i^*\star d\omega \quad \text{and} \quad \Psi^2\varphi = 0.$$

With the help of (13), this gives (9) and (10).

Turning to (11), we again let  $\omega \in \Omega^k(M)$  solve (4) for a form  $\varphi \in \Omega^k(\partial M)$ . Let  $\varepsilon \in \Omega^{k+1}(M)$  be a solution to the problem

$$\begin{cases} \Delta\varepsilon = 0 \\ i^*\varepsilon = d\varphi, \quad i^*\star\varepsilon = 0. \end{cases}$$

Then

$$(15) \quad \Phi d_{\partial}\varphi = i^*\star d\varepsilon, \quad \Psi d_{\partial}\varphi = i^*\delta\varepsilon.$$

Define  $\eta \in \Omega^{n-k-1}(M)$  by

$$(16) \quad \eta = \star d\omega - \star\varepsilon.$$

Clearly,  $\Delta\eta = 0$ . Moreover,

$$\star\eta = \star\star(d\omega - \varepsilon) = \pm(d\omega - \varepsilon),$$

so

$$i^*\star\eta = \pm i^*(d\omega - \varepsilon) = \pm(d\varphi - d\varphi) = 0.$$

Also,

$$i^*\eta = i^*\star d\omega - i^*\star\varepsilon = \Phi\varphi,$$

since  $i^*\star\varepsilon = 0$ .

Therefore,  $\eta$  solves the boundary value problem

$$\begin{cases} \Delta\eta = 0 \\ i^*\eta = \Psi\varphi, \quad i^*\star\eta = 0. \end{cases}$$

Hence,

$$(17) \quad \Phi^2\varphi = i^*\star d\eta \quad \text{and} \quad \Phi\Psi\varphi = i^*\delta\eta.$$

Using (16) we see that

$$\delta\eta = \delta\star d\omega - \delta\star\varepsilon = \pm\star dd\omega - \delta\star\varepsilon = (-1)^{k+1}\star d\varepsilon.$$

Thus,

$$i^*\delta\eta = (-1)^{k+1}i^*\star d\varepsilon,$$

which, along with (15) and (17), yields

$$\Psi\Phi\varphi = (-1)^{k+1}\Phi d_{\partial}\varphi,$$

proving (11).

Finally, (12) is proved along the same lines. From (16) we have

$$\star d\eta = \star d\star(d\omega - \varepsilon) = (-1)^{kn+1}(\delta d\omega - \delta\varepsilon).$$

Again making use of the fact that  $\delta d\omega = -d\delta\omega$ , this implies that

$$i^*\star d\omega = (-1)^{kn+1}(i^*\delta d\omega - i^*\delta\varepsilon) = (-1)^{kn}(d_{\partial}i^*\delta\omega + i^*\delta\varepsilon).$$

In turn, we can use (13) and (15) to rewrite the above formula as

$$i^*\star d\eta = (-1)^{kn}(d_{\partial}\Psi\varphi + \Psi d_{\partial}\varphi).$$

Comparing with (17), this produces the desired relation (12).  $\square$

**Remark 2.2.** The key properties of the operator  $\Lambda$  are expressed by the equalities

$$\Lambda d_{\partial} = 0, \quad d_{\partial}\Lambda = 0, \quad \text{and} \quad \Lambda^2 = 0.$$

It is straightforward to check that these equalities follow from (8) and Lemma 2.1.

### 3. RECOVERING THE BETTI NUMBERS OF $M$ FROM $\Phi$

Belishev and Sharafutdinov showed that the Betti numbers of the manifold  $M$ ,

$$\beta_k(M) = \dim H^k(M; \mathbb{R}),$$

can be recovered from the data  $(\partial M, \Lambda)$ . The proof of this fact is somewhat indirect, involving the auxiliary operator

$$(18) \quad G = \Lambda + (-1)^{kn+k+n}d_{\partial}\Lambda^{-1}d_{\partial} : \Omega^k(\partial M) \rightarrow \Omega^{n-k-1}(\partial M).$$

In contrast, it is much more straightforward to recover the Betti numbers of  $M$  from the operator  $\Phi$ .

**Theorem 1.** *Let  $\Phi_k : \Omega^k(\partial M) \rightarrow \Omega^{n-k-1}(\partial M)$  be the restriction of  $\Phi$  to  $\Omega^k(\partial M)$ . Then*

$$\beta_k(M) = \dim \ker \Phi_k.$$

The Hodge–Morrey–Friedrichs decomposition theorem [Sch95, Section 2.4] implies that

$$H^k(M; \mathbb{R}) \simeq \mathcal{H}_N^k(M),$$

where

$$\mathcal{H}_N^k(M) := \{\omega \in \Omega^k(M) : d\omega = 0, \delta\omega = 0, i^* \star \omega = 0\}$$

is the space of harmonic Neumann fields. Since harmonic forms are uniquely determined by their boundary values,  $\mathcal{H}_N^k(M) \simeq i^* \mathcal{H}_N^k(M)$ , so Theorem 1 is an immediate consequence of the following lemma.

**Lemma 3.1.** *The kernel of the operator  $\Phi_k : \Omega^k(\partial M) \rightarrow \Omega^{n-k-1}(\partial M)$  consists of the boundary traces of harmonic Neumann fields; i.e.,*

$$\ker \Phi_k = i^* \mathcal{H}_N^k(M).$$

The image of  $\Phi_k$  coincides with the subspace  $(i^* \mathcal{H}_N^k(M))^\perp \subset \Omega^{n-k-1}(\partial M)$  consisting of forms  $\psi \in \Omega^{n-k-1}(\partial M)$  satisfying

$$(19) \quad \int_{\partial M} \psi \wedge \chi = 0 \quad \forall \chi \in i^* \mathcal{H}_N^k(M).$$

In particular,  $\Phi$  is a Fredholm operator with index zero.

*Proof.* If  $\varphi \in \Omega^k(\partial M)$  such that  $\Phi_k \varphi = 0$ , then the boundary value problem

$$(20) \quad \begin{cases} \Delta\omega = 0 \\ i^* \omega = \varphi, \quad i^* \star \omega = 0, \quad i^* \star d\omega = 0 \end{cases}$$

is solvable. Using Green's formula,

$$\langle d\omega, d\omega \rangle_{L^2} + \langle \delta\omega, \delta\omega \rangle_{L^2} = \langle \Delta\omega, \omega \rangle_{L^2} + \int_{\partial M} i^*(\omega \wedge \star d\omega - \delta\omega \wedge \star \omega).$$

The right side of this equation equals zero since  $\omega$  solves the boundary value problem (20). Hence,  $\omega$  is a harmonic Neumann field since  $i^* \star \omega = 0$ , and so  $\varphi = i^* \omega \in i^* \mathcal{H}_N^k(M)$ .

The converse statement is immediate: if  $\varphi = i^* \omega$  for  $\omega \in \mathcal{H}_N^k(M)$ , then  $\omega$  solves the boundary value problem (20) and hence  $\varphi \in \ker \Phi_k$ .

On the other hand, a form  $\psi \in \Omega^{n-k-1}(\partial M)$  is in the image of  $\Phi_k$  if and only if the boundary value problem

$$\begin{cases} \Delta\omega = 0 \\ i^* \star \omega = 0, \quad i^* \star d\omega = \psi \end{cases}$$

is solvable. The defining condition (19) of  $(i^* \mathcal{H}_N^k(M))^\perp$  is precisely the necessary and sufficient condition for the solvability of this boundary value problem [Sch95, Corollary 3.4.8].  $\square$

**Corollary 3.2.** *The operator  $d_\partial \Phi^{-1}$  is well-defined on  $\text{im } \Phi_k = (i^* \mathcal{H}_N^k(M))^\perp$ ; i.e., the equation  $\Phi\varphi = \psi$  has a solution  $\varphi$  for every  $\psi \in (i^* \mathcal{H}_N^k(M))^\perp$  and  $d_\partial \varphi$  is uniquely determined by  $\psi$ .*

*Proof.* A form  $\psi \in (i^*\mathcal{H}_N^k(M))^\perp$  belongs to the range of  $\Phi$ , so the equation  $\Phi\varphi = \psi$  is solvable. If  $\Phi\varphi_1 = \Phi\varphi_2$ , then the form  $\varphi_1 - \varphi_2 \in \ker \Phi$  is closed, meaning that  $d_\partial\varphi_1 = d_\partial\varphi_2$ .  $\square$

The apparent similarity between the operator  $d_\partial\Phi^{-1}$  and the Hilbert transform  $T = d_\partial\Lambda^{-1}$  defined by Belishev and Sharafutdinov is no accident, as the following proposition demonstrates. Thus, the connection to the Poincaré duality angles of  $M$  [Sho09, Theorem 4] comes directly from the definition of  $\Phi$  (and hence  $\Pi$ ) without using  $\Lambda$  as an intermediary.

**Proposition 3.3.**  $d_\partial\Lambda^{-1} = d_\partial\Phi^{-1}$ , where the term on the right-hand side is understood to be the restriction of  $d_\partial\Phi^{-1}$  to  $\text{im } \Lambda = i^*\mathcal{H}^k(M)$ .

*Proof.* Suppose  $\varphi \in \text{im } \Lambda = i^*\mathcal{H}^k(M)$ . Then  $\varphi = i^*\omega$  for some  $\omega \in \mathcal{H}^k(M)$ . The Friedrichs decomposition says that

$$\mathcal{H}^k(M) = c\mathcal{E}\mathcal{H}^k(M) \oplus \mathcal{H}_D^k(M),$$

where

$$\begin{aligned} c\mathcal{E}\mathcal{H}^k(M) &= \{\delta\xi \in \Omega^k(M) : d\delta\xi = 0\} \\ \mathcal{H}_D^k(M) &= \{\eta \in \Omega^k(M) : d\eta = 0, \delta\eta = 0, i^*\eta = 0\}. \end{aligned}$$

Hence,

$$\omega = \delta\xi + \eta \in c\mathcal{E}\mathcal{H}^k(M) \oplus \mathcal{H}_D^k(M).$$

The form  $\xi \in \Omega^{k+1}(M)$  can be chosen such that  $\xi$  is closed,  $\Delta\xi = 0$ , and  $i^*\xi = 0$  [Sch95, p. 87, Remark 2]. Therefore,

$$\begin{cases} \Delta \star \xi = 0, \\ i^* \star (\star \xi) = 0, \\ i^* \delta \star \xi = \pm i^* \star d \star \star \xi = \pm i^* \star d\xi = 0. \end{cases}$$

This implies that  $\star\xi$  solves the boundary value problems associated to both  $\Lambda$  and  $\Phi$ , so

$$\Lambda i^* \star \xi = i^* \star d \star \xi = (-1)^{nk+1} i^* \delta \xi = (-1)^{nk+1} i^* \omega = (-1)^{nk+1} \varphi$$

and

$$\Phi i^* \star \xi = i^* \star d \star \xi = (-1)^{nk+1} i^* \delta \xi = (-1)^{nk+1} i^* \omega = (-1)^{nk+1} \varphi.$$

Hence,

$$d\Lambda^{-1}\varphi = (-1)^{nk+1} d i^* \star \xi = d\Phi^{-1} i^* \star \xi,$$

so we conclude that, indeed,  $d\Lambda^{-1} = d\Phi^{-1}$ .  $\square$

4. THE HOMOLOGY OF THE CHAIN COMPLEX  $(\Omega^*(\partial M), \Psi)$ 

We saw in Lemma 2.1 that  $\Psi^2 = 0$ , so it is natural to ask: what is the homology of the chain complex  $(\Omega^*(\partial M), \Psi)$ ?

**Theorem 2.** *For any  $0 \leq k \leq n - 1$ , if  $\Psi_k : \Omega^k(\partial M) \rightarrow \Omega^{k-1}(\partial M)$  is the restriction of  $\Psi$  to the space of  $k$ -forms on  $\partial M$ , then*

$$H_k(\Omega^*(\partial M), \Psi) = \frac{\ker \Psi_k}{\text{im } \Psi_{k+1}} \simeq H^{k+1}(M, \partial M; \mathbb{R}) \oplus H^k(M; \mathbb{R}).$$

In other words, the homology groups of  $(\Omega^*(\partial M), \Psi)$  contain the absolute cohomology groups of  $M$  in the same dimension and echoes of the relative cohomology groups of  $M$  in one higher dimension. This behavior is similar to that exhibited by the cohomology of harmonic forms studied by Cappell, DeTurck, Gluck, and Miller [CDGM06].

Since  $H^k(M; \mathbb{R}) \simeq \ker \Phi_k$  (by Theorem 1) and since it will turn out that  $\text{im } \Psi_{k+1}$  completely misses  $\ker \Phi_k$ , we can see the echo of the  $(k+1)$ st relative cohomology group of  $M$  inside the space of  $k$ -forms on  $\partial M$ .

**Corollary 3.** *The space  $\Omega^k(\partial M)$  of  $k$ -forms on  $\partial M$  contains a space isomorphic to  $H^{k+1}(M, \partial M; \mathbb{R})$  which is distinguished by the Dirichlet-to-Neumann operator  $\Pi$ . Specifically,*

$$(\ker \Psi_k / \text{im } \Psi_{k+1}) / \ker \Phi_k \simeq H^{k+1}(M, \partial M; \mathbb{R}).$$

When  $n = 2$  and  $k = 0$ , Theorem 1 and Corollary 3 imply that  $H^0(M; \mathbb{R})$  and  $H^1(M, \partial M; \mathbb{R})$  can be distinguished inside the space of functions on  $\partial M$ . Moreover, by Poincaré–Lefschetz duality,  $H^0(M; \mathbb{R}) \simeq H^2(M, \partial M; \mathbb{R})$  and  $H^1(M, \partial M; \mathbb{R}) \simeq H^1(M; \mathbb{R})$ . Since  $H^0(M, \partial M; \mathbb{R})$  and  $H^2(M; \mathbb{R})$  are both trivial, we have the following corollary.

**Corollary 4.** *All of the cohomology groups of a surface  $M$  with boundary can be realized inside the space of smooth functions on  $\partial M$ , where they can be recovered by the Dirichlet-to-Neumann operator  $\Pi$ .*

Theorem 2 will follow from Lemmas 4.1 and 4.2, which describe the kernel and image of  $\Psi$ .

**Lemma 4.1.** *If  $\Psi_k : \Omega^k(\partial M) \rightarrow \Omega^{k-1}(\partial M)$  is the restriction of  $\Psi$  to the space of  $k$ -forms on  $\partial M$ , then  $\ker \Psi_k$  is a direct sum of three spaces:*

- (i) *The pullbacks of harmonic Neumann fields*

$$i^* \mathcal{H}_N^k(M) = \ker \Phi_k.$$

- (ii) *The space*

$$\ker G_k \cap i^* \left( (\mathcal{C}^k(M))^\perp \right),$$

*which consists of the pullbacks of  $k$ -forms with conjugates on  $M$  which are perpendicular to the space of closed forms.*

- (iii) *A space isomorphic to  $H^{k+1}(M, \partial M; \mathbb{R})$ .*

The operator  $G_k$  is the restriction to  $\Omega^k(\partial M)$  of the operator  $G$  defined in (18).

**Lemma 4.2.** *The image of the operator  $\Psi_{k+1} : \Omega^{k+1}(\partial M) \rightarrow \Omega^k(\partial M)$  is precisely the space*

$$\ker G_k \cap i^* \left( (C^k(M))^\perp \right).$$

*Proof of Lemma 4.1.* Suppose  $\varphi \in \Omega^k(\partial M)$  such that  $\Psi\varphi = 0$ . Then, if  $\omega \in \Omega^k(M)$  solves the boundary value problem (4), we have that

$$(21) \quad 0 = \Psi\varphi = i^*\delta\omega.$$

Using the Hodge-Morrey decomposition of  $\Omega^k(M)$  [Sch95, Theorem 2.4.2],

$$(22) \quad \omega = \delta\xi + \kappa + d\zeta \in c\mathcal{E}_N^k(M) \oplus \mathcal{H}^k(M) \oplus \mathcal{E}_D^k(M),$$

where

$$\begin{aligned} c\mathcal{E}_N^k(M) &= \{\omega \in \Omega^k(M) : \omega = \delta\xi \text{ for some } \xi \in \Omega^{k+1}(M) \text{ with } i^*\star\xi = 0\} \\ \mathcal{H}^k(M) &= \{\omega \in \Omega^k(M) : d\omega = 0, \delta\omega = 0\} \\ \mathcal{E}_D^k(M) &= \{\omega \in \Omega^k(M) : \omega = d\zeta \text{ for some } \zeta \in \Omega^{k-1}(M) \text{ with } i^*\zeta = 0\}. \end{aligned}$$

Equations (21) and (22) imply that

$$(23) \quad 0 = i^*\delta\omega = i^*\delta(\delta\xi + \kappa + d\zeta) = i^*\delta d\zeta.$$

Since  $\delta d\zeta$  is co-exact and since the space of co-exact  $k$ -forms is precisely the orthogonal complement of the space of  $k$ -forms satisfying a Dirichlet boundary condition, (23) implies that  $\delta d\zeta = 0$ . Hence,  $d\zeta$  is co-closed—but  $\mathcal{E}_D^k(M)$  is precisely the orthogonal complement of the space of co-closed  $k$ -forms, so it follows that  $d\zeta = 0$ .

Therefore,

$$\omega = \delta\xi + \kappa$$

is co-closed. Since both  $\omega$  and  $\delta\xi \in c\mathcal{E}_N^k(M)$  satisfy a Neumann boundary condition,  $\kappa$  must be a harmonic Neumann field. Moreover, since both  $\omega$  and  $\kappa$  are harmonic, it follows that  $\delta\xi$  is harmonic. Hence,

$$\omega = \delta\xi + \kappa \in (c\mathcal{E}_N^k(M) \cap \ker \Delta) \oplus \mathcal{H}_N^k(M)$$

and so

$$(24) \quad \varphi = i^*\omega \in i^*(c\mathcal{E}_N^k(M) \cap \ker \Delta) + i^*\mathcal{H}_N^k(M).$$

Conversely, forms in this space are clearly in the kernel of  $\Psi$ .

In (24) the sum of spaces is not, *a priori*, direct, but directness of the sum follows immediately from the fact that harmonic forms are uniquely determined by their boundary values [Sch95, Theorem 3.4.10].

The term  $i^*\mathcal{H}_N^k(M) = \ker \Phi_k$  in (24) is exactly the space described in (i), so the lemma will follow from showing that  $i^*(c\mathcal{E}_N^k(M) \cap \ker \Delta)$  is the direct sum of the spaces described in (ii) and (iii).

Suppose, then, that  $\varphi \in i^*(c\mathcal{E}_N^k(M) \cap \ker \Delta)$ ; i.e., that  $\omega = \delta\xi$ . Since  $0 = \Delta\omega = \Delta\delta\xi$ , we know that

$$0 = (d\delta + \delta d)\delta\xi = \delta d\delta\xi,$$

so  $d\delta\xi$  is co-closed, meaning that  $d\delta\xi \in \mathcal{H}^{k+1}(M)$ ; specifically,  $d\delta\xi \in \mathcal{E}\mathcal{H}^{k+1}(M)$ . On the other hand, for any  $d\gamma \in \mathcal{E}\mathcal{H}^{k+1}(M)$ , there is a unique choice of primitive  $\gamma$  that is in  $c\mathcal{E}_N^k(M) \cap \ker \Delta$ . Hence,

$$c\mathcal{E}_N^k(M) \cap \ker \Delta \simeq \mathcal{E}\mathcal{H}^{k+1}(M).$$

In turn, since forms in  $c\mathcal{E}_N^k(M) \cap \ker \Delta$  are uniquely determined by their pullbacks to the boundary, this implies that

$$i^*(c\mathcal{E}_N^k(M) \cap \ker \Delta) \simeq \mathcal{E}\mathcal{H}^{k+1}(M).$$

Applying the Hodge star to the space  $c\mathcal{E}_N^k(M) \cap \ker \Delta$  yields Cappell, DeTurck, Gluck, and Miller's space  $\text{EHarm}^{n-k}$ . Thinking in those terms,  $\delta\xi \in c\mathcal{E}_N^k(M)$  is a harmonic, co-exact form, but the primitive  $\xi$  is not necessarily harmonic. There are two possibilities:

**Case 1:** If  $\xi$  is harmonic, then

$$0 = \Delta\xi = (d\delta + \delta d)\xi = d\delta\xi + \delta d\xi,$$

meaning that  $d\delta\xi = -\delta d\xi$  is both exact and co-exact. Since  $\Delta\delta\xi = 0$ , this means that  $\delta\xi$  has a conjugate form (in the sense of [BS08, Section 5]). This implies that  $i^*\delta\xi \in \ker G_k$  [BS08, Theorem 5.1]. Since  $\delta\xi$  is orthogonal to the space of closed  $k$ -forms on  $M$ , we have

$$\varphi = i^*\delta\xi \in \ker G_k \cap i^*\left((\mathcal{C}^k(M))^\perp\right),$$

which is the space in (ii).

Conversely, if  $\varphi \in \ker G_k \cap i^*\left((\mathcal{C}^k(M))^\perp\right)$ , then  $\varphi = i^*\delta\xi$  for some  $\delta\xi \in c\mathcal{E}_N^k(M)$  which has a conjugate form. This implies that  $d\delta\xi$  is both exact and co-exact, and it is straightforward to check that  $\xi$  can be chosen to be harmonic.

**Case 2:** If  $\xi$  is not harmonic, then it belongs to the space

$$\mathcal{N}^k := \{\delta\xi \in c\mathcal{E}_N^k(M) \cap \ker \Delta : \Delta\xi \neq 0\}.$$

This space is isomorphic to  $H^{k+1}(M, \partial M; \mathbb{R})$  [CDGM06, Lemma 3], and so  $i^*\mathcal{N}^k$  is the space given in (iii).

The directness of the sum

$$\left(\ker G_k \cap i^*\left((\mathcal{C}^k(M))^\perp\right)\right) + i^*\mathcal{N}^k$$

again follows from the fact that harmonic forms are uniquely determined by their boundary values.  $\square$

We can now determine the image of  $\Psi_{k+1}$ .

*Proof of Lemma 4.2.* Suppose  $\vartheta \in \Omega^k(\partial M)$  such that  $\vartheta = \Psi\varphi$  for some  $\varphi \in \Omega^{k+1}(\partial M)$ . If  $\omega \in \Omega^{k+1}(M)$  solves the boundary value problem (4), then  $\vartheta = \Psi\varphi = i^*\delta\omega$ .

Since  $\omega$  satisfies a Neumann boundary condition,

$$\delta\omega \in c\mathcal{E}_N^k(M).$$

Moreover, since  $\Delta$  commutes with the co-differential,

$$\Delta\delta\omega = \delta\Delta\omega = 0,$$

and so

$$\delta\omega \in c\mathcal{E}_N^k(M) \cap \ker \Delta.$$

Since  $\omega$  is itself harmonic, this is precisely the situation described in Case 1 of the proof of Lemma 4.1, so

$$\vartheta = i^*\delta\omega \in \ker G_k \cap i^*\left((\mathcal{C}^k(M))^\perp\right).$$

Conversely, if  $\vartheta = i^*\delta\zeta$  for  $\delta\zeta \in c\mathcal{E}_N^k(M) \cap \ker \Delta$  with  $\zeta$  harmonic, then

$$\Delta\zeta = 0 \quad \text{and} \quad i^*\star\zeta = 0,$$

so  $\vartheta = i^*\delta\zeta = \Psi i^*\zeta$  is in the image of  $\Psi$ .  $\square$

**Corollary 4.3.**

$$\ker \Phi_k \subset \ker \Psi_k \quad \text{and} \quad \text{im } \Psi_k \subset \text{im } \Phi_{n-k}.$$

*Proof.* The fact that  $\ker \Phi_k \subset \ker \Psi_k$  is an immediate consequence of Lemma 4.1.

Now, suppose  $\varphi \in \text{im } \Psi_k$ . Then, by Lemma 4.2,  $\varphi \in \ker G_{k-1}$ , meaning  $\varphi = i^*\omega$  for  $\omega \in \Omega^{k-1}(M)$  satisfying

$$\Delta\omega = 0, \quad \delta\omega = 0, \quad \text{and} \quad d\omega = \star d\eta$$

for some  $\eta \in \Omega^{n-k-1}(M)$  with  $\Delta\eta = 0$  and  $\delta\eta = 0$  [BS08, Theorem 5.1].

Therefore, for any  $\lambda_N \in \mathcal{H}_N^{n-k}(M)$ ,

$$(25) \quad \int_{\partial M} \varphi \wedge i^*\lambda_N = \pm \int_{\partial M} i^*\omega \wedge i^*(\star\lambda_N) = \pm [\langle d\omega, \star\lambda_N \rangle_{L^2(M)} - \langle \omega, \delta\star\lambda_N \rangle_{L^2(M)}]$$

by Green's formula. The second term on the right hand side vanishes since  $\lambda_N$  is closed, while the first is equal to

$$(26) \quad \langle \star d\eta, \star\lambda_N \rangle_{L^2(M)} = \langle d\eta, \lambda_N \rangle_{L^2(M)} = 0.$$

The first equality above is due to the fact that  $\star$  is an isometry and the second follows because  $\mathcal{H}_N^{n-k}(M)$  is orthogonal to the space of exact forms on  $M$ .

Putting (25) and (26) together shows that

$$\int_{\partial M} \varphi \wedge i^*\lambda_N = 0$$

for any  $\lambda_N \in \mathcal{H}_N^{n-k}(M)$ , so Lemma 3.1 implies that  $\varphi \in \text{im } \Phi_{n-k}$ , as desired.  $\square$

5. COCHAIN MAPS AND THE ADJOINT OF  $\Psi$ 

Since  $\Psi$  is a chain map whose homologies are interesting, it seems natural to try to find associated cochain maps and compute their cohomologies. In fact, there are two such maps,

$$\tilde{\Psi} := (-1)^{k(n-1)} \star_{\partial} \Psi \star_{\partial} \quad \text{and} \quad \Theta := (-1)^{(k+1)(n-1)} \Phi \Psi \Phi.$$

By definition both are maps  $\Omega^k(\partial M) \rightarrow \Omega^{k+1}(\partial M)$ .

5.1. **The operator  $\tilde{\Psi}$ .** The fact that  $\tilde{\Psi}^2 = 0$  is immediate:

$$\tilde{\Psi}^2 = \pm \star_{\partial} \Psi \star_{\partial} \star_{\partial} \Psi \star_{\partial} = \pm \star_{\partial} \Psi^2 \star_{\partial} = 0,$$

since  $\Psi^2 = 0$ .

Let  $\tilde{\Psi}^k$  be the restriction of  $\tilde{\Psi}$  to  $\Omega^k(\partial M)$ . Since  $\star_{\partial}$  is an isomorphism,

$$\ker \tilde{\Psi}^k \simeq \ker \Psi_{n-k-1} \quad \text{and} \quad \text{im } \tilde{\Psi}^{k-1} \simeq \text{im } \Psi_{n-k},$$

and so

$$(27) \quad H^k(\Omega^*(\partial M), \tilde{\Psi}) \simeq H_{n-k-1}(\Omega^*(\partial M), \Psi).$$

Thus, we can use Theorem 2 to determine the cohomology groups of  $\tilde{\Psi}$ .

**Proposition 5.1.** *The cohomology groups of the cochain complex  $(\Omega^*(\partial M), \tilde{\Psi})$  are*

$$H^k(\Omega^*(\partial M), \tilde{\Psi}) \simeq H^{n-k}(M; \mathbb{R}) \oplus H^{n-k-1}(M, \partial M; \mathbb{R})$$

The obvious guess, suggested by experience with  $\Lambda$  and by the duality given in (27), is that  $\tilde{\Psi}$  is the adjoint of  $\Psi$ .

**Proposition 5.2.**  *$\tilde{\Psi}$  is the adjoint of  $\Psi$ .*

*Proof.* The proof follows along similar lines to the proof that  $\Lambda^* = \star_{\partial} \Lambda \star_{\partial}$  [BS08, p. 132].

Let  $\varphi \in \Omega^k(\partial M)$  and  $\psi \in \Omega^{n-k}(\partial M)$ . Suppose  $\omega \in \Omega^k(M)$  solves the boundary value problem (4) and that  $\eta \in \Omega^{n-k}(M)$  solves the equivalent boundary value problem for  $\psi$ .

The key step is to show that

$$(28) \quad (-1)^{k+1} \int_{\partial M} \varphi \wedge \Psi \psi = (-1)^{kn+n+1} \int_{\partial M} \psi \wedge \Psi \varphi.$$

Provided this is true, we can re-write the above equation as

$$(-1)^{kn+k+1} \langle \varphi, \star_{\partial} \Psi \psi \rangle_{L^2(\partial M)} = - \langle \psi, \star_{\partial} \Psi \varphi \rangle_{L^2(\partial M)}$$

or, equivalently,

$$\langle \varphi, \star_{\partial} \Psi \psi \rangle_{L^2(\partial M)} = (-1)^{k(n-1)} \langle \psi, \star_{\partial} \Psi \varphi \rangle_{L^2(M)}.$$

Letting  $\psi = \star_{\partial} \psi'$ , this becomes

$$\langle \psi, \star_{\partial} \Psi \star_{\partial} \psi' \rangle_{L^2(\partial M)} = (-1)^{k(n-1)} \langle \star_{\partial} \psi', \star_{\partial} \Psi \varphi \rangle_{L^2(\partial M)} = (-1)^{k(n-1)} \langle \psi', \Psi \varphi \rangle_{L^2(\partial M)},$$

since  $\star_{\partial}$  is an isometry. Therefore,

$$\Psi^* = (-1)^{k(n-1)} \star_{\partial} \Psi \star_{\partial} = \tilde{\Psi},$$

as desired.

To prove (28) we note that, by Green's formula,

$$\begin{aligned} \int_{\partial M} \varphi \wedge \Psi \psi &= \int_{\partial M} i^* \omega \wedge i^* \delta \eta = (-1)^{n(k+1)+n+1} \int_{\partial M} i^* \omega \wedge i^* (\star d \star \eta) \\ (29) \qquad \qquad \qquad &= (-1)^{kn+1} (\langle d\omega, d \star \eta \rangle_{L^2(M)} - \langle \omega, \delta d \star \eta \rangle_{L^2(M)}). \end{aligned}$$

Notice that

$$-\langle \omega, \delta d \star \eta \rangle_{L^2(M)} = \langle \omega, d \delta \star \eta \rangle_{L^2(M)}$$

since  $0 = \star \Delta \eta = \Delta \star \eta = d \delta \star \eta + \delta d \star \eta$ . In turn,

$$\langle \delta \omega, \delta \star \eta \rangle_{L^2(M)} = \langle \omega, d \delta \star \eta \rangle_{L^2(M)} - \int_{\partial M} i^* \delta \star \eta \wedge i^* \star \omega.$$

Since  $i^* \star \omega = 0$ , the second term on the right hand side vanishes. Therefore, we can re-write (29) as

$$(30) \quad \int_{\partial M} \varphi \wedge \Psi \psi = (-1)^{kn+1} (\langle d\omega, d \star \eta \rangle_{L^2(M)} + \langle \delta \omega, \delta \star \eta \rangle_{L^2(M)}).$$

Completely analogous reasoning yields the expression

$$(31) \quad \int_{\partial M} \psi \wedge \Psi \varphi = (-1)^{kn+n+1} (\langle d\eta, d \star \omega \rangle_{L^2(M)} + \langle \delta \eta, \delta \star \omega \rangle_{L^2(M)})$$

Therefore, (28) follows from (30) and (31) because

$$\begin{aligned} \langle d\omega, d \star \eta \rangle_{L^2(M)} &= \langle \star d\omega, \star d \star \eta \rangle_{L^2(M)} = (-1)^{k(n+1)} \langle \delta \star \omega, \delta \eta \rangle_{L^2(M)} \\ \langle \delta \omega, \delta \star \eta \rangle_{L^2(M)} &= \langle \star \delta \omega, \star \delta \star \eta \rangle_{L^2(M)} = (-1)^{k(n+1)} \langle d \star \omega, d \eta \rangle_{L^2(M)} \end{aligned}$$

(the first equality in each line is due to the fact that  $\star$  is an isometry).  $\square$

**5.2. The operator  $\Theta$ .** There are several different equivalent ways of expressing the operator  $\Theta = (-1)^{(k+1)(n+1)} \Phi \Psi \Phi$ . Using (9),

$$(32) \quad \Theta = (-1)^{(k+1)(n+1)} \Phi \Psi \Phi = (-1)^{kn} d_{\partial} \Phi^2.$$

On the other hand, using (11),

$$(33) \quad \Theta = (-1)^{(k+1)(n+1)} \Phi \Psi \Phi = (-1)^{n(k+1)} \Phi^2 d_{\partial}.$$

Finally, combining (12) with (33) yields

$$(34) \quad \Theta = (-1)^{n(k+1)} \Phi^2 d_{\partial} = (d_{\partial} \Psi + \Psi d_{\partial}) d_{\partial} = d_{\partial} \Psi d_{\partial}.$$

This last expression makes it clear that  $\Theta$  is a cochain map:

$$\Theta^2 = d_{\partial} \Psi d_{\partial} d_{\partial} \Psi d_{\partial} = 0.$$

**Proposition 5.3.** *The cohomology of the cochain complex  $(\Omega^*(\partial M), \Theta)$  is given, up to isomorphism, by*

$$H^k(\Omega^*(\partial M), \Theta) \simeq H^{k+1}(M, \partial M; \mathbb{R}) \oplus H^k(M; \mathbb{R}).$$

Notice that  $(\Omega^*(\partial M), \Theta)$  has the same cohomology as  $(\Omega^*(\partial M), \tilde{\Psi})$ .

We omit the proof of Proposition 5.3, which is somewhat long and technical, though not particularly difficult. Two perhaps surprising consequences are:

- (i) Since  $\Theta$  has the same cohomology as  $\tilde{\Psi}$ , the homology of  $\Psi$  can be completely recovered from that of  $\Theta$ . However, by (34),  $\Theta = d_{\partial}\Psi d_{\partial}$ , so pre- and post-composing  $\Psi$  by  $d_{\partial}$  does not change the (co)homology.
- (ii) By (32) and (33),

$$\Theta = \pm d_{\partial}\Phi^2 = \pm\Phi^2 d_{\partial}.$$

Hence, the homology of  $\Psi$  is completely determined by the operator  $\Phi$ , and the results of Corollaries 3 and 4 depend only on  $\Phi$ . In that spirit, the following is a restatement of the  $k = 0$  case of Corollary 3.

**Corollary 5.** *A copy of the cohomology group  $H^{n-1}(M; \mathbb{R})$  is distinguished by the operator  $\Phi$  inside  $\Omega^0(\partial M)$ , the space of smooth functions on  $\partial M$ . Specifically,*

$$\ker(d_{\partial}\Phi^2) / \ker \Phi \simeq H^{n-1}(M; \mathbb{R}).$$

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SOBOLEV INSTITUTE OF MATHEMATICS

*E-mail address:* [sharaf@math.nsc.ru](mailto:sharaf@math.nsc.ru)

*URL:* <http://www.math.nsc.ru/~sharafutdinov/>

DEPARTMENT OF MATHEMATICS, HAVERFORD COLLEGE

*E-mail address:* [cshonkwi@haverford.edu](mailto:cshonkwi@haverford.edu)

*URL:* <http://www.haverford.edu/math/cshonkwi/>