AN INTEGRAL FORMULA FOR $\bar{\mu}_{123}$

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Abstract. In [1], Polyak demonstrated that Milnor's μ invariant for 3-component links can be computed combinatorially. In [3] Polyak and Viro showed how to compute the Casson invariant of a knot as the degree of a map. Here, we emulate the style of [3] and use the results of [1] to compute the μ invariant as the sum of degrees of maps.

1. Gauss Diagrams

In [1], Polyak demonstrated that Milnor's μ invariant for 3-component links can be computed combinatorially by counting certain subdiagrams of a Gauss diagram for the link. Let us first recall how this works.

For a given n-component link $L \subset \mathbb{R}^3$, consider a diagram D of L as an immersion of n circles in the plane. At each double point we assign a sign, corresponding to the sign of the crossing. The corresponding *Gauss diagram* G of L is a collection of n immersing circles with the two preimages of each double point connected by an oriented chord. Each chord is decorated with the sign associated with the corresponding crossing and oriented so that it points from the overcrossing point to the undercrossing point. For two examples borrowed from Polyak, see Figure 1.

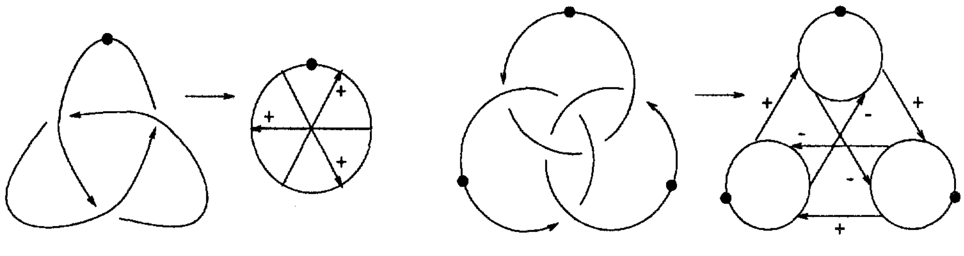


Figure 1. Gauss diagrams for the trefoil and the Borromean rings

A *based* Gauss diagram is a Gauss diagram with a marked point (basepoint) on each circle which is distinct from the endpoints of the chords. We can define the *sign* of a subdiagram of a Gauss diagram G as the product of the signs of the chords of the subdiagram.

For a given Gauss diagram G and some other Gauss diagram A, we use the notation $\langle A, G\rangle$ to denote the algebraic number of subdiagrams of G of type A. Although such numbers depend, *a priori*, on the choice of diagram for a link, Polyak and Viro have demonstrated that several are independent of the choice of diagram and thus are knot or link invariants. For example, in [2], Polyak and Viro show the following theorem:

Theorem 1.1 (Polyak-Viro). *If G is any based Gauss diagram of a knot K, then*

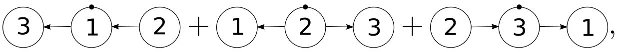
$v_2(K) = \Big\langle$  $, G\Big\rangle$

where $v_2(K)$ is the Vassiliev invariant of degree 2 which evaluates to 0 on the unknot and 1 on the trefoil.

Remark 1.2. The invariant v_2 in Theorem 1.1 is often called the *Casson invariant.*

The same paper gives a formula for the invariant $\bar{\mu}_{123}$ of a 3-component link, but the following formulation from [1] is simpler:

Theorem 1.3 (Polyak-Viro). *For any based diagram G of an ordered 3-component link L,*

(1) $\bar{\mu}_{123}(L) = \Big\langle$  $, G\Big\rangle$

Nice as this formula is, our goal is, of course, to derive an integral formula for Milnor's μ invariant. Perhaps surprisingly, (1) can be converted into an integral formula.

2. The Casson invariant as the degree of a map

The model for all arguments that some knot or link invariant can be computed as the degree of a map is, of course, the argument that the linking number of two oriented, disjoint circles in $L_1, L_2 \subset \mathbb{R}^3$ is equal to the degree of the map $\phi : L_1 \times L_2 \to S^2$ given by

$$\phi(x, y) = \frac{x - y}{|x - y|}$$

In [3] Polyak and Viro showed how to compute the Casson invariant of a knot as the degree of a map. They do so by defining a map whose degree is equal to $\Big\langle$  $, G\Big\rangle$, which, as we saw in Theorem 1.1, is equal to the Casson invariant of a knot K with Gauss diagram G. In this section we outline the general strategy they employ, with the goal of adapting it to a strategy for interpreting the μ invariant as the degree of a map and, ultimately, deriving an integral formula for it.

In order to express the Casson invariant as the degree of a map, we first need to define a configuration space $\mathcal{C}$ to be the domain of the map. The construction of $\mathcal{C}$ proceeds in several steps. First, for a knot $K \subset \mathbb{R}^3$ with a base point $* \in K$, let C_X denote the space of 4-tuples $(x_1, x_2, x_3, x_4) \in K^4$

of points ordered such that, when following the orientation of K, the points occur in the order $*, x_1, x_2, x_3, x_4, *$. Let C_X^0 be the subspace of C_X defined by the inequalities $* \neq x_1 \neq x_2 \neq x_3 \neq x_4 \neq *$ and with the orientation induced by the orientation of K and the order of the points. Define the map $\phi_X^0 : C_X^0 \to S^2 \times S^2$ given by

$$(x_1, x_2, x_3, x_4) \mapsto \left(\frac{x_1 - x_3}{|x_1 - x_3|}, \frac{x_4 - x_2}{|x_4 - x_2|} \right),$$

which extends to a map ϕ_X on all of C_X by continuity.

In the above, the notation X is used to stand for the picture  precisely because C_X and ϕ_X were constructed so that the preimage of the point

$$(S, S) = (\text{south pole, south pole}) \in S^2 \times S^2$$

under ϕ_X consists of configurations of points corresponding to subdiagrams of the Gauss diagram isomorphic to . Hence, the Casson invariant v_2 can be seen as the local degree of the map ϕ_X.

Unfortunately, this local degree does not automatically extend to a global degree for a number of reasons. First, C_X is not a closed manifold but, rather, a manifold with boundary and corners. Second, the map ϕ_X fails to be proper at the point (S, S) when there are triple points or self tangencies in the projection of K to the xy-plane. Although such cases are not generic, a generic isotopy involves situations where the projection is not generic (e.g. during a Reidemeister III move). Dealing with these problems requires considerable delicacy and the construction of several other spaces to be glued to or removed from the space C_X, but in the end the Casson invariant can indeed be computed as the degree of a map. In fact, Polyak and Viro give two different integral formulas for the Casson invariant, the first of which we reproduce here:

Theorem 2.1 (Polyak-Viro). *Let*

$$\Omega = \frac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

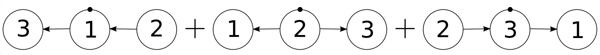
Then, if $K \subset \mathbb{R}^3$ is a knot and if we denote by C_V^0 the space of 3-tuples $(x_1, x_2, x_3) \in K^3$ ordered in the natural way defined by the orientation of K with $* \neq x_1 \neq x_2 \neq x_3 \neq *$, then

$$\begin{aligned} v_2(K) \quad = \quad & \int_{C_X^0} \Omega(x_1 - x_3) \wedge \Omega(x_4 - x_2) \\ & + \frac{1}{2} \int_{C_V^0} \int_{t \in (0,1)} \Omega(x_1 - x_2) \wedge \Omega((x_2 - x_3) + t(x_1 - x_2)) \\ & + \frac{1}{2} \int_{C_V^0} \int_{t \in (0,1)} \Omega(x_2 - x_3) \wedge \Omega((x_1 - x_2) + t(x_2 - x_3)) \\ & + \frac{1}{2} \int_{C_V^0} \int_{t \in (-\infty,0) \cup (1,\infty)} \Omega(x_3 - x_1) \wedge \Omega((x_1 - x_2) + t(x_3 - x_2)) \end{aligned} \tag{2}$$

3. μ AS THE DEGREE OF A MAP

Now we want to apply the strategy outlined in Section 2 to the case of Milnor's μ invariant. First, recall (1):

$$\bar{\mu}_{123}(L) = \Big\langle$$



$$, G\Big\rangle$$

What we want to do is to choose a configuration space corresponding to each term in this sum, define a map on that space such that the local degree of the map is equal to the corresponding term in (1) and then find some way of gluing the configuration spaces together to get a map on a compact manifold. Unfortunately, base point problems will plague us a bit.

In fact, we will choose a configuration space corresponding to each term in the sum and to each of the three possible ways of putting the base point on one of the three components. The three spaces corresponding to the placement of the base point on, e.g., the first component will then compute something which looks like one of the terms in (1) but with the base point on the first component.

As it turns out, we will be able to glue the three spaces corresponding to the placement of the base point on the ith component to another, auxiliary configuration space. The result of these gluings will be a closed manifold $\mathcal{C}_i$ on which we will have a well-defined, continuous map ϕ_i whose degree will, after dividing by 6, be congruent modulo the greatest common divisor of the pairwise linking numbers to one of the terms in (1) (in particular, the term with its base point on the correct component).

Thus, by adding the degrees of the ϕ_i, we will recover the full right hand side of (1) modulo the greatest common divisor of the pairwise linking numbers. Since $\bar{\mu}_{123}$ is only defined modulo this gcd, this will give $\bar{\mu}_{123}$ as the sum of degrees of maps (see Theorem 3.3).

3.1. **The basic configuration spaces.** Suppose $L \subset \mathbb{R}^3$ is a 3-component link with components L_1, L_2, L_3 which are oriented, closed, disjoint, smooth curves. Moreover, assume

$$L_1 = \{x(s)\}, \qquad L_2 = \{y(t)\}, \qquad L_3 = \{z(u)\}$$

for parameters $s, t, u \in S^1$. In fact, we will consider the s-circle, the t-circle and the u-circle as three separate circles, S^1_1, S^1_2 and S^1_3, respectively. We will consider these circles as being unmarked unless decorated with a $*$ as in S^1_{1*}; when there is a marked point, the marked point of S^1_{i*} will be denoted $*_i$.

Now, define the configuration spaces

$$C^0_{\Lambda_{11}} := \{(s_1, s_2, t, u) \in S^1_{1*} \times S^1_{1*} \times S^1_2 \times S^1_3 | *_1 \neq s_1 \neq s_2 \neq *_1\},$$

$$C^0_{\Lambda_{12}} := \{(s, t_1, t_2, u) \in S^1_{1*} \times S^1_2 \times S^1_2 \times S^1_3 | *_1 \neq s, t_1 \neq t_2\}$$

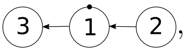
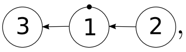
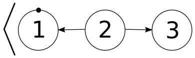
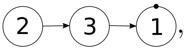
and

$$C^0_{\Lambda_{13}} := \{(s, t, u_1, u_2) \in S^1_{1*} \times S^1_2 \times S^1_3 \times S^1_3 | *_1 \neq s, u_1 \neq u_2\}$$

where s_1 and s_2 are oriented so that traveling counterclockwise from s_1 to s_2 one does not pass through the basepoint.

Now, for each $i = 1, 2, 3$, we define $\phi^0_{1i} : C^0_{\Lambda_{1i}} \to S^2 \times S^2$ where

$$\begin{aligned}
\phi^0_{11} :&(s_1, s_2, t, u) \mapsto \left(\frac{z(u) - x(s_1)}{|z(u) - x(s_1)|}, \frac{x(s_2) - y(t)}{|x(s_2) - y(t)|} \right) \\
\phi^0_{12} :&(s, t_1, t_2, u) \mapsto \left(\frac{x(s) - y(t_1)}{|x(s) - y(t_1)|}, \frac{z(u) - y(t_2)}{|z(u) - y(t_2)|} \right) \\
\phi^0_{13} :&(s, t, u_1, u_2) \mapsto \left(\frac{z(u_1) - y(t)}{|z(u_1) - y(t)|}, \frac{x(s) - z(u_2)}{|x(s) - z(u_2)|} \right).
\end{aligned}$$

By construction, we see that the preimage of (south pole, south pole) $\in S^2 \times S^2$ under ϕ^0_{11} is equal to $\Big\langle$  $, G\Big\rangle$ where G is a Gauss diagram for L. In other words, $\Big\langle$  $, G\Big\rangle$ is equal to the local degree of ϕ^0_{11}. Similarly, $\Big\langle$  $, G\Big\rangle$ is equal to the local degree of ϕ^0_{12} and $\Big\langle$  $, G\Big\rangle$ is equal to the local degree of ϕ^0_{13}. These last two are not exactly what we want to count, but we will deal with that problem later.

Of course, here we run into the same problems that Polyak and Viro do. First, our $C^0_{\Lambda_{1i}}$ are not compact, so there is no global degree. To fix this, consider $C^0_{\Lambda_{11}}$ as a subspace of $S^1_1 \times S^1_1 \times S^1_2 \times S^1_3$ and take the closure $C_{\Lambda_{11}} := \overline{C^0_{\Lambda_{11}}}$. This is now a manifold with boundary and corners to which we can extend ϕ^0_{11} to a map $\phi_{11} : C_{\Lambda_{11}} \to S^2 \times S^2$ by continuity.

The various strata of the boundary of $C_{\Lambda_{11}}$ occur when, for some subset $A \subset \{*_1, s_1, s_2\}$, the elements of A are equal. Denote the stratum corresponding to $A = \{*_1, s_1, s_2\}$ by $\Sigma\Lambda_{11_{*12}}$ and similarly for other choices of A. Now, the stratum $\Sigma\Lambda_{11_{*12}}$ of the boundary is of codimension 2 and, thus, its image under ϕ_1 will be inessential in $S^2 \times S^2$. The strata $\Sigma\Lambda_{1_{*1}}$ and $\Sigma\Lambda_{1_{*2}}$ are of codimension 1, so we can't just ignore them. Instead, we increase the codimension of the image using the same trick as Polyak and Viro: we exile the base point to infinity.

In particular, this means that, if we think of L as living in S^3, we stereographically project from the image of $*_1$. Moreover, we may perform a small isotopy so that $L_1 = \{x(s)\}$ coincides with a geodesic in a small neighborhood of $x(*_1)$ so that the image in $\mathbb{R}^3$ lives on a straight line outside some ball. Finally, we may as well assume that this straight line is the y-axis.

Now, when $\infty = *_1 = s_1$,

$$\phi_1(\Sigma\Lambda_{11_{*1}} \subset \{(0, \pm 1, 0)\} \times S^2,$$

which has codimension 2 in $S^2 \times S^2$ and so is inessential for degree-of-map considerations. A similar argument shows that $\phi_1(\Sigma\Lambda_{11_{*2}}$ is inessential.

Therefore, the only boundary stratum of $C_{\Lambda_{11}}$ that we have to worry about is $\Sigma\Lambda_{11_{12}}$

Similar arguments to the above give compactifications of $C^0_{\Lambda_{12}}$ and $C^0_{\Lambda_{13}}$ on which we only need to worry about the boundary strata $\Sigma\Lambda_{12_{12}}$ and $\Sigma\Lambda_{13_{12}}$.

Moreover, the $\Sigma\Lambda_{1i_{12}}$ are all homeomorphic to the space $C^0_{V_1} := \{(s,t,u) \in S^1_{1*} \times S^1_2 \times S^1_3 | *_1 \neq s\}$ via the maps $\xi_{1i} : \Sigma\Lambda_{1i_{12}} \to C^0_{V_1}$ given by

$$\begin{aligned}\xi_{11} &: (s,s,t,u) \mapsto (s,t,u)\\ \xi_{12} &: (s,t,t,u) \mapsto (s,t,u)\\ \xi_{13} &: (s,t,u,u) \mapsto (s,t,u).\end{aligned}$$

Completely analogously, we will define $C_{\Lambda_{2i}}$ and ϕ_{2i} for $i = 1,2,3$ and $C_{\Lambda_{3j}}$ and ϕ_{3j} for $j = 1,2,3$. The local degrees of the ϕ_{2i} count $\left\langle \textcircled{3} \leftarrow \textcircled{1} \leftarrow \overset{\bullet}{\textcircled{2}}, G \right\rangle$, $\left\langle \textcircled{3} \leftarrow \overset{\bullet}{\textcircled{1}} \leftarrow \textcircled{2}, G \right\rangle$ and $\left\langle \overset{\bullet}{\textcircled{2}} \rightarrow \textcircled{3} \rightarrow \textcircled{1}, G \right\rangle$ for $i = 1,2$ and 3, respectively and the local degrees of the ϕ_{3j} count $\left\langle \overset{\bullet}{\textcircled{3}} \leftarrow \textcircled{1} \leftarrow \textcircled{2}, G \right\rangle$, $\left\langle \textcircled{1} \leftarrow \textcircled{2} \rightarrow \overset{\bullet}{\textcircled{3}}, G \right\rangle$ and $\left\langle \textcircled{2} \rightarrow \overset{\bullet}{\textcircled{3}} \rightarrow \textcircled{1}, G \right\rangle$ for $j = 1,2$ and 3, respectively.

In each case we recover one term that we want and two that we don't and for each j $C_{\Lambda_{ij}}$ has one principal boundary face which is homeomorphic to the boundary face of the spaces corresponding to other choices of j.

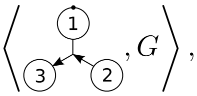
3.2. **An auxiliary configuration space.** Ideally, we would like to glue the $C_{\Lambda_{1i}}$ together to get a closed space on which is defined a map ϕ_1 which extends each of the ϕ_{1i}. Unfortunately, we can't quite do this. Instead, we must introduce a new configuration space, which we will call C_{Y_1}. We will define C_{Y_1} as we did the $C_{\Lambda_{ij}}$, starting first with an open configuration space and then compactifying.

To that end, define $C^0_{Y_1} := \{(s,t,u,r) \in S^1_{1*} \times S^1_2 \times S^1_3 \times \mathbb{R}^3 | *_1 \neq s, r \neq x(s), r \neq y(t), r \neq z(u)\}$ (here we suppose that the base point has already been exiled to infinity as in the construction of the $C_{\Lambda_{1i}}$). Moreover, define the map $\phi^0_{Y_1} : C^0_{Y_1} \to S^2 \times S^2 \times S^2$ by

$$\phi^0_{Y_1} : (s,t,u,r) \mapsto \left(\frac{x(s) - r}{|x(s) - r|}, \frac{r - y(t)}{|r - y(t)|}, \frac{z(u) - r}{|z(u) - r|} \right).$$

Remark 3.1. Since the base point has already been exiled to infinity, the image of the obvious boundary given by $*_1 = s$ has codimension 2 and, therefore, is inessential.

Note that the inverse image of the point (south pole, south pole, south pole) $\in S^2 \times S^2 \times S^2$ is equal to

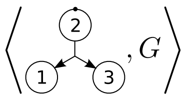


which, since it looks like an inverted Y, is why we've chosen to call this space $C^0_{Y_1}$.

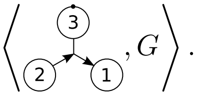
Since $C^0_{Y_1}$ is an open manifold, we would like to embed it in some compact space and take the closure. Unfortunately, although $C^0_{Y_1} \subset \overline{S^1_{1*}} \times S^1_2 \times S^1_3 \times S^3$ which is compact (we use the notation $\overline{S^1_{1*}}$ to indicate that s may coincide with $*_1$), $\phi^0_{Y_1}$ does not admit a continuous extension to this space.

We resolve this issue by using the same trick Polyak and Viro do in compactifying their space C_Y: let Γ be the graph of $\phi^0_{Y_1}$ in $\overline{S^1_{1*}} \times S^1_2 \times S^1_3 \times S^3 \times S^2 \times S^2 \times S^2$ and let C_{Y_1} be its closure in this space. Let B_{Y_1} be the image of C_{Y_1} under the natural projection $\pi : \overline{S^1_{1*}} \times S^1_2 \times S^1_3 \times S^3 \times S^2 \times S^2 \times S^2 \to \overline{S^1_{1*}} \times S^1_2 \times S^1_3 \times S^3$; then π identifies Γ with $C^0_{Y_1}$ and, with this identification in mind, the natural projection $C_{Y_1} \to S^2 \times S^2 \times S^2$ extends the original map $\phi^0_{Y_1}$. We denote this extension by ϕ_{Y_1}.

In completely analogous fashion, we may define C_{Y_2}, ϕ_{Y_2}, C_{Y_3} and ϕ_{Y_3} so that $\phi^{-1}_{Y_2}(\text{south pole}, \text{south pole}, \text{south pole})$ counts



and $\phi^{-1}_{Y_3}(\text{south pole}, \text{south pole}, \text{south pole})$ counts



Each of the C_{Y_i} is a compact manifold with boundary. We analyze the boundary next.

3.3. **The principal faces of C_{Y_i}.** Let $p = (s, t, u, r, v_1, v_2, v_3) \in C_{Y_1} \backslash \Gamma$ be a point on the boundary of C_{Y_1}. Since $\pi(p) = (s, t, u, r) \in B_{Y_1}$ belongs to the boundary of $C^0_{Y_1}$, either $s = *_1 = \infty$ or r coincides with one of the points $x(s), y(t)$ or $z(u)$. Denote these last three by ΣY_{11}, ΣY_{12} and ΣY_{13}, respectively. Since we already noted that the image of the boundary corresponding to $*_1 = s$ is inessential, the ΣY_{1i} are the only boundary faces we will worry about.

In fact, these boundary faces are easy to deal with, as the following lemma suggests:

Lemma 3.2. *For $i = 1, 2, 3$, the map $\eta_{1i} : \Sigma Y_{1i} \to C^0_{V_1} \times S^2$ given by*

$$(s, t, u, r, v_1, v_2, v_3) \mapsto ((s, t, u), v_i)$$

is a homeomorphism of degree $(-1)^i$ with respect to the naturally induced orientations on ΣY_{1i} and $C^0_{V_1} \times S^2$.

In turn, all of what we've said applies equally well to C_{Y_2} and C_{Y_3}, with suitable changes of indices.

3.4. **Gluing the pieces together.** Now that we have all the pieces in place, the obvious thing to do is to glue $C_{\Lambda_{ij}} \times S^2$ to C_{Y_i} for each $i, j = 1, 2, 3$. We know that $\Sigma\Lambda_{ij_{12}} \times S^2 \simeq C^0_{V_i} \times S^2 \simeq \Sigma Y_{ij}$, so this should be feasible. Unfortunately, we need to make these identifications in such a way that the ϕ_{ij} and ϕ_{Y_i} extend to a continuous map ϕ. To see why this requires some delicacy, fix r, s and t and consider the image of $\{(s, t, u, r, v_1, v_2, v_3)\}$ under ϕ_{Y_1} as r coincides with $x(s)$, $y(t)$ or $z(u)$. When $r = x(s)$, the image is equal to

$$S^2 \times \left\{ \left(\frac{r - y(t)}{|r - y(t)|}, \frac{z(u) - r}{|z(u) - r|} \right) \right\}$$

with the S^2 term as the first factor. When $r = z(u)$, the image is equal to

$$\left\{ \left(\frac{x(s) - r}{|x(s) - r|}, \frac{r - y(t)}{|r - y(t)|} \right) \right\} \times S^2$$

with the S^2 term as the third factor. Finally, when $r = y(t)$, the image is homeomorphic to

$$\left\{ \left(\frac{x(s) - r}{|x(s) - r|}, \frac{z(u) - r}{|z(u) - r|} \right) \right\} \times S^2$$

but with the S^2 term as the second factor.

These complications force us to consider $C_{Y_1} \times S_3$ and $C_{\Lambda_{1i}} \times S_3$ for $i = 1, 2, 3$ where S_3 is the symmetric group on 3 letters. For $\sigma \in S_3$, let $\bar{\sigma} : S^2 \times S^2 \times S^2 \to S^2 \times S^2 \times S^2$ be the re-ordering of the factors of $S^2 \times S^2 \times S^2$ corresponding to σ.

Now, we define the space $\mathcal{C}_1$ as the quotient space of

$$\left(\bigcup_{i=1}^{3} (C_{\Lambda_{1i}} \times S^2) \cup C_{Y_1} \right) \times S_3$$

determined by the following identifications:

- $\Sigma\Lambda_{11_{12}} \times S^2 \times \sigma$ is identified with $\Sigma Y_{11} \times \sigma \circ (13)$ via $(\xi_{11} \times \mathrm{id}_{S^2}) \circ \eta_{11}^{-1}$.
- $\Sigma\Lambda_{12_{12}} \times S^2 \times \sigma$ is identified with $\Sigma Y_{12} \times \sigma \circ (23)$ via $(\xi_{12} \times \mathrm{id}_{S^2}) \circ \eta_{12}^{-1}$.
- $\Sigma\Lambda_{13_{12}} \times S^2 \times \sigma$ is identified with $\Sigma Y_{13} \times \sigma \circ (12)$ via $(\xi_{13} \times \mathrm{id}_{S^2}) \circ \eta_{13}^{-1}$.

It is now easy to check that

$$\bar{\sigma} \circ (\phi_{1i} \times \mathrm{id}_{S^2}) : C_{\Lambda_{1i}} \times S^2 \times \sigma \to S^2 \times S^2 \times S^2$$

and

$$\bar{\sigma} \circ \phi_{Y_1} : C_{Y_1} \times \sigma \to S^2 \times S^2 \times S^2$$

give rise to a continuous map $\phi_1 : \mathcal{C}_1 \to S^2 \times S^2 \times S^2$ (see Figure 2).

Similar constructions give rise to the spaces $\mathcal{C}_2$ and $\mathcal{C}_3$ and maps ϕ_2 and ϕ_3. Each of the $\mathcal{C}_i$ is a closed, connected manifold and the result of all of this effort is the following theorem:

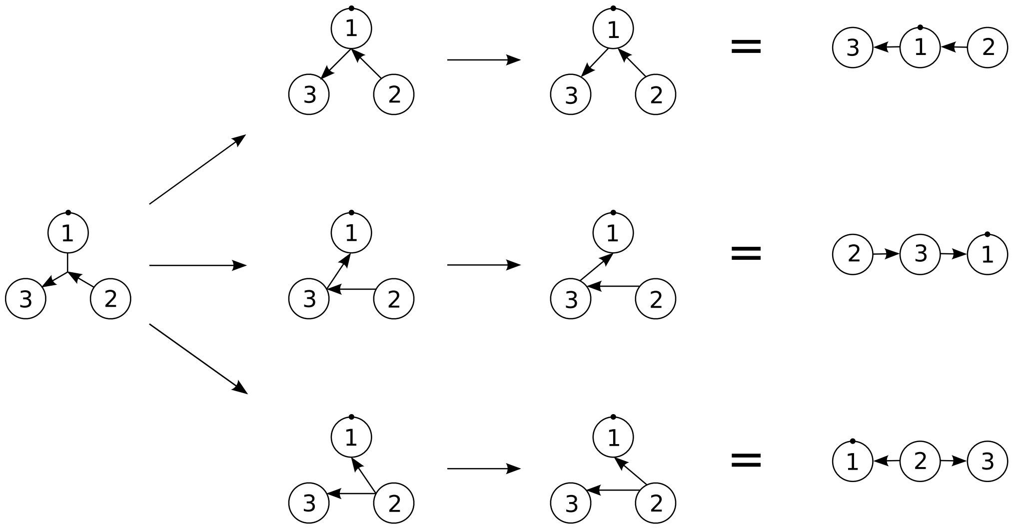


FIGURE 2. Identifying the boundaries of C_{Y_1} and $C_{\Lambda_{1i}}$

Theorem 3.3. *The space $\mathcal{C}_i$ has a well-defined fundamental class $[\mathcal{C}_i] \in H_6(\mathcal{C}_i)$. Moreover,*

$$\frac{1}{6}\left[\deg(\phi_i)+\deg(\phi_2)+\deg(\phi_3)\right] \equiv \bar{\mu}_{123}(L) \pmod{\delta} \tag{3}$$

where δ is the greatest common divisor of the pairwise linking numbers.

Proof. The first sentence follows directly from our construction.

To evaluate the degrees of the ϕ_i, assume our link L is in general position with respect to the vertical projection. To compute the degree of ϕ_i, count the algebraic number of points in the preimage of a regular value $p \in S^2 \times S^2 \times S^2$ close to (south pole, south pole, south pole). Specifying to $i = 1$, those preimages of p which lie in each of the six copies of $\bigcup_i (C_{\Lambda_{1i}} \times S^2)$ count

$$\left\langle \textcircled{3}\leftarrow\dot{\textcircled{1}}\leftarrow\textcircled{2} + \textcircled{2}\rightarrow\textcircled{3}\rightarrow\dot{\textcircled{1}} + \dot{\textcircled{1}}\leftarrow\textcircled{2}\rightarrow\textcircled{3}, G\right\rangle .$$

Moreover,

$$\left\langle \textcircled{2}\rightarrow\textcircled{3}\rightarrow\dot{\textcircled{1}}, G\right\rangle = Lk(L_2,L_3)\cdot Lk(L_1,L_3)$$

and

$$\left\langle \dot{\textcircled{1}}\leftarrow\textcircled{2}\rightarrow\textcircled{3}, G\right\rangle = Lk(L_1,L_2)\cdot Lk(L_2,L_3).$$

Therefore, the number of preimages of p in the $C_{\Lambda_{1i}}$ is congruent to $6 \cdot \left\langle \textcircled{3}\leftarrow\dot{\textcircled{1}}\leftarrow\textcircled{2}, G\right\rangle$ modulo $6\cdot\delta$. Moreover, the p has no preimages in any of the copies of C_{Y_1}, because such a preimage would correspond to a triple point in the vertical projection and we assumed L is in general position with respect to the vertical projection. Thus, we see that

$$\deg(\phi_1) \equiv 6\cdot\left\langle \textcircled{3}\leftarrow\dot{\textcircled{1}}\leftarrow\textcircled{2}, G\right\rangle \pmod{6\cdot\delta}.$$

Similarly, the degrees of ϕ_2 and ϕ_3 are congruent to $6\cdot\left\langle \textcircled{1} \leftarrow \overset{\bullet}{\textcircled{2}} \rightarrow \textcircled{3}, G\right\rangle$ and $6\cdot\left\langle \textcircled{2} \rightarrow \overset{\bullet}{\textcircled{3}} \rightarrow \textcircled{1}, G\right\rangle$ modulo $6\cdot\delta$, respectively. (3) then follows directly from Theorem 1.3. □

References

[1] Michael Polyak: On Milnor's triple linking number. *C.R. Math. Acad. Sci. Paris*, 325(1):77–82, Jul 1997. [doi:10.1016/S0764-4442(97)83937-7].

[2] Michael Polyak and Oleg Viro: Gauss diagram formulas for Vassiliev invariants. *Int. Math. Res. Not.*, 1994(11):445–453, Nov 1994. [doi:10.1155/S1073792894000486].

[3] Michael Polyak and Oleg Viro: On the Casson knot invariant. *J. Knot Theory Ramifications*, 10(5):711–738, Jun 2001. [doi:10.1142/S0218216501001116, arXiv:math/9903158v1].

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