# Research Statement
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### 1. Introduction

My research is largely focused on the interplay between geometry, topology, differential forms, and vector fields. Different approaches to these ideas have led me into the areas of differential geometry, inverse problems, knot theory, and contact geometry, but in each case my underlying motivation is to understand some manifestation of the connection between geometry and topology.

Some of the questions that have guided my work are:

- It is well-known that the cohomology of a manifold is represented by differential forms, but how much of the geometry of the manifold do these representatives of cohomology encode? Can they tell us how "close" the manifold is to being complete?
- How much of the geometry and topology of a manifold can be recovered from physically meaningful boundary data? Can the manifold be completely reconstructed from this data?
- Very few useful topological invariants of vector fields are known: can we construct new ones that provide lower bounds for the field energy?
- How can we detect geometric properties of Legendrian knots?

These are broad questions, but in each case I have been able to give some answers. The first two are related by new invariants of Riemannian manifolds with boundary and discussed in §2. The third—discussed in §3—leads to interesting problems in knot theory, homotopy theory, and Fourier analysis. My approach to the fourth involves non-commutative algebraic invariants and is discussed in §4. Finally, some specific directions for future research are laid out in §5.

### 2. Geometry, topology, and differential forms

The connection between topology and differential forms goes back at least to Helmholtz [30], who of course used the language of vector fields. The modern formulation is due to de Rham [16], who showed that the cohomology of a compact, oriented, smooth manifold is encoded in the space of differential forms on the manifold. Specifically,

$$H^p(M;\mathbb{R}) \cong \mathcal{C}^p(M)/\mathcal{E}^p(M),$$

where $\mathcal{C}^p(M)$ is the space of closed $p$-forms on $M$ and $\mathcal{E}^p(M)$ is the space of exact $p$-forms.

When there is an inner product on the space $\Omega^p(M)$ of $p$-forms on $M$, it's natural to expect that $H^p(M;\mathbb{R})$ should be isomorphic to the orthogonal complement of $\mathcal{E}^p(M)$ inside $\mathcal{C}^p(M)$. If the manifold $M$ is Riemannian, then the Riemannian metric induces an inner product on $\Omega^p(M)$ and this expected behavior holds. Specifically, the Hodge decomposition theorem [32, 33, 36] says that, on a closed, oriented, smooth Riemannian manifold $M$,

$$H^p(M;\mathbb{R}) \cong \mathcal{H}^p(M),$$

where $\mathcal{H}^p(M)$ is the space of $p$-forms which are both closed and co-closed (dubbed "harmonic fields" by Kodaira).

The situation is more complicated when the manifold has boundary because $\mathcal{H}^p(M)$ is infinite-dimensional, but Morrey [50] and Friedrichs [25] (and, at least implicitly, Duff and Spencer [20]) generalized the Hodge decomposition theorem to this setting; here

$$H^p(M;\mathbb{R}) \cong \mathcal{H}^p_N(M) \quad \text{and} \quad H^p(M,\partial M;\mathbb{R}) \cong \mathcal{H}^p_D(M),$$

where $\mathcal{H}^p_N(M)$ is the space of harmonic fields satisfying a Neumann boundary condition and $\mathcal{H}^p_D(M)$ is the space of harmonic fields satisfying a Dirichlet boundary condition.

2.1. **Poincaré duality angles.** As one would expect, the spaces $\mathcal{H}^p_N(M)$ and $\mathcal{H}^p_D(M)$ meet only at the zero form, but, as pointed out by DeTurck and Gluck [17], they are not generally orthogonal. In [59] I discuss new invariants of Riemannian manifolds with boundary, called *Poincaré duality angles*, which measure the relative positions of these concrete realizations of cohomology. Specifically, the Poincaré duality angles for $M$ in dimension $p$ are the non-trivial principal angles between $\mathcal{H}^p_N(M)$ and $\mathcal{H}^p_D(M)$.

Since $H^p(M;\mathbb{R}) = H^p(M,\partial M;\mathbb{R})$ when $M$ is closed (and, likewise, the Neumann and Dirichlet boundary conditions are vacuous for closed manifolds), we can say, somewhat euphemistically, that the Poincaré duality angles of a closed manifold are all zero. Since the Poincaré duality angles of a manifold with boundary can never be zero, this suggests that the Poincaré duality angles measure how "close" a manifold is to being closed.

To test this hypothesis, I computed the Poincaré duality angles of some interesting manifolds which are intuitively close to being closed. For example, consider the complex projective space $\mathbb{CP}^n$ with its usual Fubini-Study metric and define the manifold

$$M_r = \mathbb{CP}^n - B_r(x)$$

obtained by removing a ball of radius $r$ centered at the point $x \in \mathbb{CP}^n$.

**Theorem 2.1** ([59]). *For $1 \leq k \leq n-1$, there is a non-trivial Poincaré duality angle $\theta_r^{2k}$ between the concrete realizations of $H^{2k}(M_r;\mathbb{R})$ and $H^{2k}(M,\partial M;\mathbb{R})$ given by*

$$\cos\theta_r^{2k} = \frac{1-\sin^{2n} r}{\sqrt{(1+\sin^{2n} r)^2 + \frac{(n-2k)^2}{k(n-k)}\sin^{2n} r}}.$$

Notice that, as $r \to 0$, the angle $\theta_r^{2k} \to 0$, and as $r$ approaches its maximum value of $\pi/2$, the angle $\theta_r^{2k} \to \pi/2$. Theorem 2.1 immediately generalizes to other nontrivial $D^2$-bundles over $\mathbb{CP}^{n-1}$, so this asymptotic behavior of the Poincaré duality angles is not dependent on being able to close up the manifold by capping off with a ball.

Similar results hold in more general settings where the tubular neighborhood of a submanifold is removed. These examples provide evidence that the Poincaré do reflect geometric properties of the manifold and lead to Conjecture 5.1, stated below.

2.2. **Connections to inverse problems.** In the same paper [59], I prove a connection between the Poincaré duality angles and the Dirichlet-to-Neumann operator which arises in the problem of Electrical Impedance Tomography (EIT). The EIT problem, which was first posed by Calderón [10], is to determine the conductivity inside a region $M$ from knowledge of the voltage-to-current map on the boundary of the region. Calderón was thinking in terms of geoprospecting when he posed the problem, but the EIT problem is also of significant importance in medical imaging [34], where the goal is to detect tumors or other abnormalities inside the body without having to cut the patient open.

Stating things in terms of conductivities and voltage-to-current maps is somewhat awkward mathematically, but Lee and Uhlmann [44] showed that recovering the conductivity from the voltage-to-current map is equivalent to recovering a Riemannian metric on the region $M$ from the classical Dirichlet-to-Neumann map $\Lambda_{\text{cl}}$.

A prototypical theorem for this problem is the following.

**Theorem 2.2** (Lee and Uhlmann [44])**.** *If $M^n$ is a simply-connected, compact, real-analytic, geodesically convex Riemannian manifold with boundary and $n \geq 3$, then $(\partial M, \Lambda_{\text{cl}})$ determines the Riemannian metric on $M$ up to isometry. If "real-analytic" is replaced with "smooth", then $(\partial M, \Lambda_{cl})$ determines the $C^\infty$-jet of the metric at the boundary of $M$.*

Generalizations of Theorem 2.2 are given by Lassas and Uhlmann [43] and Lassas, Taylor, and Uhlmann [42].

Belishev and Sharafutdinov [7] generalized the Dirichlet-to-Neumann map to differential forms on $\partial M$ as the operator $\Lambda : \Omega^p(\partial M) \to \Omega^{n-p-1}(\partial M)$ given by

$$\Lambda\varphi = i^* \star \omega,$$

where $i : \partial M \to M$ is the identical inclusion and $\omega \in \Omega^p(M)$ is a (non-unique) solution to the boundary value problem

$$\Delta\omega = 0, \quad i^*\omega = \varphi, \quad i^*\delta\omega = 0.$$

On three-dimensional manifolds, the Dirichlet-to-Neumann operator for 1-forms is closely related to the problem of magnetostatics [5].

Belishev and Sharafutdinov showed that the Dirichlet-to-Neumann data $(\partial M, \Lambda)$ determines the additive cohomology structure of $M$.

**Theorem 2.3** (Belishev and Sharafutdinov [7])**.** *Define the operator*

$$G = \Lambda + (-1)^{pn+p+n} d\Lambda^{-1} d.$$

*Then the dimension of the image of $G$ is equal to the dimension of the absolute cohomology group $H^{n-p-1}(M; \mathbb{R})$.*

Using the *Hilbert transform* $T = d\Lambda^{-1}$ I showed that rather more is true: the data $(\partial M, \Lambda)$ not only determines the cohomology groups of $M$, but detects the relative positions of the concrete realizations of the absolute and relative cohomology groups. Specifically:

**Theorem 2.4** ([59])**.** *If $\theta_1, \ldots, \theta_k$ are the Poincaré duality angles of $M$ in dimension $p$, then the quantities*

$$(-1)^{pn+n+p} \cos^2 \theta_i$$

*are the nontrivial eigenvalues of the operator $T^2$.*

After showing that $(\partial M, \Lambda)$ determines the additive cohomology of $M$, Belishev and Sharafutdinov posed the following question:

> Can the multiplicative structure of cohomologies be recovered from our data $(\partial M, \Lambda)$? Till now, the authors cannot answer the question.

I gave a partial answer to this question.

**Theorem 2.5** ([59])**.** *The mixed cup product*

$$\cup : H^p(M;\mathbb{R}) \times H^q(M,\partial M;\mathbb{R}) \to H^{p+q}(M,\partial M;\mathbb{R})$$

*is completely determined by the data $(\partial M, \Lambda)$ when the relative class is restricted to come from the boundary subspace of $H^q(M,\partial M;\mathbb{R})$.*

My approach has recently been adapted by Al-Zamil and Montaldi [1] to partially recover the cup product in the context of equivariant cohomology.

Belishev and Sharafutdinov's operator $\Lambda$ is actually only one of two attempts to generalize the classical Dirichlet-to-Neumann map to differential forms. Joshi and Lionheart [35] defined an operator $\Pi : \Omega^p(M)|_{\partial M} \to \Omega^{n-p-1}(M)|_{\partial M}$ and showed that the data $(\partial M, \Pi)$ determines the $C^\infty$-jet of the Riemannian metric at the boundary. Krupchyk, Lassas, and Uhlmann [41] have recently extended this result to show that $(\partial M, \Pi)$ determines the manifold up to isometry when $M$ is real-analytic.

The operators $\Lambda$ and $\Pi$ are similar, but do not appear to be equivalent. One advantage of $\Lambda$, especially for the task of recovering topological data, is that it is defined invariantly. In our work [58], Sharafutdinov and I give an invariant definition of $\Pi$ in terms of two auxiliary operators

$$\Phi : \Omega^p(\partial M) \to \Omega^{n-p-1}(\partial M) \quad \text{and} \quad \Psi : \Omega^p(\partial M) \to \Omega^{p-1}(\partial M).$$

We can easily show that $\Lambda$ is determined by $\Phi$ and $\Psi$, so it makes sense to regard the operator $\Pi$ as the "complete" Dirichlet-to-Neumann operator for differential forms.

Whereas Belishev and Sharafutdinov's proof of Theorem 2.3 was somewhat circuitous, Sharafutdinov and I show that it is very straightforward to recover the cohomology groups of $M$ from $\Phi$ (and hence from $\Pi$).

**Theorem 2.6** (with Sharafutdinov [58])**.** *The pth absolute cohomology group of $M$ is given, up to isomorphism, by*

$$H^p(M;\mathbb{R}) \cong \ker \Phi.$$

The operator $\Psi$ turns out to be a chain map, and the homology of the chain complex $(\Omega^*(\partial M), \Psi)$ is given in terms of a mixture of the absolute and relative cohomology groups of $M$.

**Theorem 2.7** (with Sharafutdinov [58])**.** *For any $0 \le p \le n-1$,*

$$H_p(\Omega^*(\partial M), \Psi) \cong H^{p+1}(M,\partial M;\mathbb{R}) \oplus H^p(M;\mathbb{R}).$$

This implies that the space of $p$-forms on $\partial M$ contains an "echo" of the $(p+1)$st relative cohomology group of $M$ which is detected by the Dirichlet-to-Neumann operator. Theorem 2.7 also implies that copies of all of the cohomology groups of a surface with boundary live inside the space of smooth functions on the boundary and that these copies are distinguished by $\Pi$.

## 3. Higher helicities for vector fields

Let $\Omega \subset \mathbb{R}^3$ be a compact region (i.e. the closure of an open set). If $V$ is a vector field on $\Omega$, then the *helicity* of $V$ is defined to be

$$\mathrm{Hel}(V) = \frac{1}{4\pi} \int_{\Omega \times \Omega} V(x) \times V(y) \cdot \frac{x-y}{|x-y|^3} dx\, dy.$$

Helicity was first defined by Woltjer [64], who was studying the Crab Nebula and noticed that the helicity of a magnetic field remains constant as the field evolves according to the equations of ideal magnetohydrodynamics, and that it provides a lower bound for the field energy during such evolution. To date, helicity was the only known topological invariant of vector fields providing such bounds.

The fact that helicity provides a lower bound for the field energy has made it an extremely important tool in applications to plasma physics and fluid dynamics, but there are circumstances when the helicity vanishes yet the field cannot evolve to have arbitrarily small energy. This suggests a problem proposed by Arnol′d and Khesin [3] regarding "higher helicities" for divergence-free vector fields. In their words:

> The dream is to define such a hierarchy of invariants for generic vector fields such that, whereas all the invariants of order $\leq k$ have zero value for a given field and there exists a nonzero invariant of order $k+1$, this nonzero invariant provides a lower bound for the field energy.

3.1. **Integral formulas for link-homotopy invariants.** Given a two-component link $L \subset \mathbb{R}^3$ with components $X = \{x(s) : s \in S^1\}$ and $Y = \{y(t) : t \in S^1\}$, Gauss showed [28] that the linking number of $X$ with $Y$ is given by

$$Lk(X,Y) = \frac{1}{4\pi}\int_{S^1\times S^1} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \frac{x(s)-y(t)}{|x(s)-y(t)|^3}\, ds\, dt.$$

There is no mistaking the analogy between the definition of helicity and Gauss's linking integral, and indeed Arnol′d showed that the helicity of a vector field is the average asymptotic linking number of its orbits [2]. This suggests that higher helicities may be analogous to higher order linking invariants and that a first step to defining higher helicities is to find integral formulas for linking invariants.

In an effort to distinguish linking phenomena from knotting phenomena, Milnor considered the notion of *link-homotopy*: a link-homotopy is a homotopy of the link in which distinct components must stay disjoint but each component is allowed to pass through itself (and so, in particular, each component can be unknotted). Milnor defined link-homotopy invariants called $\bar{\mu}$-invariants [47, 48] and showed that these invariants completely classify three-component links up to link-homotopy.

Specifically, a three-component link is determined up to link-homotopy by the three pairwise linking numbers $p$, $q$, and $r$, and by Milnor's triple linking number $\bar{\mu}_{123}$, often denoted $\mu$. The triple linking number is an integer when the pairwise linking numbers all vanish; otherwise it is only an integer modulo $\gcd(p,q,r)$.

There is a long history of finding integral formulas for Milnor's triple linking number (starting with Massey [45, 46]) and of using these integrals to try to define higher-order helicities (e.g. Monastyrsky and Retakh [49] and Berger [8, 9], among many others). The principal sources for these formulas are Massey triple products in cohomology and Chern–Simons theory. A common feature of these integral formulas is that choices must be made to fix the domain of integration and the value of the integrand.

The key feature of the Gauss linking integral is that it is *geometric* in the sense that the integrand is invariant under orientation-preserving isometries of $\mathbb{R}^3$. In physical terms, this means the integrand is coordinate-independent and thus (potentially) physically meaningful. The goal in our work [18, 19] was to find a similarly geometric integral formula for Milnor's triple linking number. This is related to work I did with Vela-Vick [60], where we found Gauss-type linking integrals for higher-dimensional links in the $n$-sphere.

To find a geometric integral formula for the triple linking number, we interpreted the link-homotopy invariants of a three-component link in the 3-sphere as homotopy invariants of an associated characteristic map to a configuration space. Configuration spaces come into the story as follows. If $L$ is a link in a 3-manifold $M$ with $n$ components $X, Y, \ldots$ parametrized by $x = x(s)$, $y = y(t), \ldots$ for $s, t, \ldots \in S^1$, then there is a natural evaluation map

$$e_L : T^n \to \mathrm{Conf}_n M, \quad (s, t, \ldots) \mapsto (x(s), y(t), \ldots)$$

from $T^n = S^1 \times \cdots \times S^1$ to the configuration space $\mathrm{Conf}_n M$ of ordered $n$-tuples of distinct points in $M$. Since link-homotopies of $L$ become homotopies of $e_L$, the assignment $L \mapsto e_L$ induces a map

$$e : \mathfrak{L}_n(M) \to [T^n, \mathrm{Conf}_n M]$$

from the set $\mathfrak{L}_n(M)$ of link-homotopy classes of $n$-component links in $M$ to the set $[T^n, \mathrm{Conf}_n M]$ of homotopy classes of maps from $T^n$ to $\mathrm{Conf}_n M$.

We can think of the map $e$ as defining a representation from the world of link-homotopy to the world of homotopy, and one can ask whether or not this representation is faithful. When $n = 2$ and $M = \mathbb{R}^3$, the answer is "yes", and this leads directly to the Gauss linking integral.

For three-component links in $S^3$, the configuration space $\mathrm{Conf}_3 S^3$ deformation retracts to $S^3 \times S^2$; projecting to the $S^2$ factor yields the *characteristic map* $g_L : T^3 \to S^2$. Maps from the 3-torus to the 2-sphere were classified up to homotopy by Pontryagin [53, 54]. A complete set of invariants is given by the degrees $p$, $q$, and $r$ of the restrictions to the 2-dimensional subtori, and by the residue class $\nu$ of one further integer modulo *twice* the greatest common divisor of $p$, $q$, and $r$.

**Theorem 3.1** (with DeTurck, et al. [18, 19])**.** *Let $L$ be a three-component link in $S^3$. Then the pairwise linking numbers $p$, $q$, and $r$ of $L$ are equal to the degrees of its characteristic map $g_L$ on the two-dimensional subtori, while twice Milnor's $\mu$-invariant for $L$ is equal to Pontryagin's $\nu$-invariant for $g_L$ modulo $2\gcd(p, q, r)$.*

Theorem 3.1 implies that the representation $e : \mathfrak{L}_3(S^3) \to [T^3, \mathrm{Conf}_3 S^3]$ is faithful; in turn, this implies that the representation $e : \mathfrak{L}_3(\mathbb{R}^3) \to [T^3, \mathrm{Conf}_3 \mathbb{R}^3]$ is also faithful.

When the pairwise linking numbers are all zero, the $\mu$- and $\nu$-invariants are ordinary integers. Since the $\nu$-invariant is essentially a generalized Hopf invariant, this allows us to adapt J. H. C. Whitehead's integral formula for the Hopf invariant [63] to get an integral formula for the triple linking number.

**Theorem 3.2** (with DeTurck, et al. [18, 19])**.** *If the pairwise linking numbers of a three-component link $L$ in $S^3$ are all zero, then Milnor's $\mu$-invariant of $L$ is given by each of the following equivalent formulas*

$$\begin{aligned}\mu(L) &= \frac{1}{2}\int_{T^3} \delta(\varphi * \omega_L) \wedge \omega_L \\ &= -\frac{1}{2}\int_{T^3 \times T^3} \mathbf{v}_L(\mathbf{x}) \times \mathbf{v}_L(\mathbf{y}) \cdot \nabla_{\mathbf{x}} \varphi(\mathbf{x} - \mathbf{y}) d\mathbf{x}\, d\mathbf{y} \\ &= 8\pi^3 \sum_{\mathbf{n} \in \mathbb{Z}^3 - \mathbf{0}} \mathbf{a_n} \times \mathbf{b_n} \cdot \frac{\mathbf{n}}{|\mathbf{n}|^2},\end{aligned}$$

*where $\varphi$ is the fundamental solution of the scalar Laplacian on the 3-torus, $\omega_L$ and $\mathbf{v}_L$ are the characteristic 2-form and vector field of $L$, and $\mathbf{a_n}$ and $\mathbf{b_n}$ are the real and imaginary parts of the Fourier coefficients $\mathbf{c_n}$ of $\omega_L$ and $\mathbf{v}_L$.*

Though I omit them here, we give explicit formulas for $\varphi$, $\omega_L$, $\mathbf{v}_L$, and $\mathbf{c_n}$. Note that the Fourier series interpretation of $\mu$ is particularly amenable to calculation by computer.

Komendarczyk [38, 39] used Theorem 3.2 and asymptotic techniques similar to Arnol′d's to define a higher-order helicity for vector fields constrained to unlinked domains and to derive an energy bound. This restriction on the domain is somewhat unnatural, but there appear to be severe technical obstructions to pushing through the asymptotic analysis without it.

3.2. **Koschorke's conjecture.** Cohen, Komendarczyk, and I have focused on the question of whether the representation $e : \mathcal{L}_n(\mathbb{R}^3) \to [T^n, \mathrm{Conf}_n\mathbb{R}^3]$ is faithful for general $n$. Let $\mathcal{B}_n(\mathbb{R}^3) \subset \mathcal{L}_n(\mathbb{R}^3)$ be the set of link-homotopy classes of links which become trivial when *any* component is deleted (so-called *Brunnian links*).

**Theorem 3.3** (Koschorke [40])**.** *The map $e : \mathcal{B}_n(\mathbb{R}^3) \to [T^n, \mathrm{Conf}_n\mathbb{R}^3]$ is injective, with $\overline{\mu}$-invariants of a link $L$ corresponding to the homotopy periods of $e_L$.*

In fact, the above theorem also holds in higher dimensions. Koschorke's natural conjecture was that the full representation $e : \mathcal{L}_n(\mathbb{R}^3) \to [T^n, \mathrm{Conf}_n\mathbb{R}^3]$ is also faithful. As noted above, the $n = 3$ case of this conjecture follows immediately from Theorem 3.1.

In our work [15], Cohen, Komendarczyk, and I build on Koschorke's result in two ways. First, we give an explicit description of the image of the Brunnian links inside $[T^n, \mathrm{Conf}_n\mathbb{R}^3]$. To set things up, note that there is a projection $\mathrm{Conf}_n\mathbb{R}^3 \to \mathrm{Conf}_{n-1}\mathbb{R}^3$ given by deleting the $i$th coordinate. There are $n$ possible choices of coordinates to delete, so these $n$ projections produce the combined projection

$$p_n : \mathrm{Conf}_n\mathbb{R}^3 \to \left(\mathrm{Conf}_{n-1}\mathbb{R}^3\right)^n .$$

This map is not a fibration, but we can construct the homotopy fibre $X_n$ of $p_n$. When $L$ is Brunnian the map $e_L$ lifts to a map $\widetilde{e_L} : T^n \to X_n$ which corresponds to an element of $\pi_n(X_n) \cong \bigoplus_{(n-1)!}\mathbb{Z}$. This group is generated by $(n-1)$-fold iterated Whitehead products of maps $\beta_{n,i} : S^2 \to \mathrm{Conf}_n\mathbb{R}^3$ without repeats in the index $i$.

**Theorem 3.4** (with Cohen and Komendarczyk [15])**.** *The set $e(\mathcal{B}_n(\mathbb{R}^3)) \subset [T^n, Conf_n\mathbb{R}^3]$ is in bijective correspondence with $\pi_n(X_n)$, with the correspondence given by the Milnor $\bar{\mu}$-invariants. Specifically, if we write $[\widetilde{e_L}] \in \pi_n(X_n)$ in terms of the generators, the coefficient of $[\beta_{n,\sigma(1)}, \ldots, \beta_{n,\sigma(n-1)}]$ is equal to $\bar{\mu}_{\sigma(1),\ldots,\sigma(n-1),n}(L)$.*

Second, we show that the representation $e$ is faithful on a much larger class of $n$-component links than the Brunnian links. To do so, we make use of the group of $n$-component homotopy string links $\mathcal{H}(n)$, which Habegger and Lin [29] defined and showed is isomorphic to the semi-direct product $\mathcal{K}(n-1) \ltimes \mathcal{H}(n-1)$. Here $\mathcal{K}(n-1)$ is the subgroup of $n$-component string links which become trivial when the $n$th strand is deleted; this group is isomorphic to the reduced free group on $n-1$ letters.

We find an injection of $\mathcal{K}(n-1)$ into $[T^{n-1}, \Omega\mathrm{Conf}_n\mathbb{R}^3]$, where $\Omega\mathrm{Conf}_n\mathbb{R}^3$ is the based loop space of the configuration space; this, combined with a Barratt-Puppe argument, leads to the following generalization of Koschorke's result.

**Theorem 3.5** (with Cohen and Komendarczyk [15]). *The representation $e$ is injective on the subset of $\mathcal{L}(n)$ consisting of $n$-component links which become trivial when* some *strand is deleted.*

3.3. **Iterated helicities.** Cantarella and I [11] have taken an alternative approach to defining higher helicities. If $V$ is a vector field on a compact region $\Omega \subset \mathbb{R}^3$, then helicity can be interpreted as an $L^2$ inner product:

$$\mathrm{Hel}(V) = \int_\Omega V \cdot BS(V)\,\mathrm{dvol},$$

where $BS$ is the Biot-Savart operator given by

$$BS(V)(y) = \frac{1}{4\pi}\int_\Omega V(x) \times \frac{y-x}{|y-x|^3}\,\mathrm{dvol}_x.$$

If $V$ is a current distribution, then $BS(V)$ is the induced magnetic field. When $V$ is divergence-free and tangent to the boundary, $\nabla \times BS(V) = V$ in $\Omega$.

When $V$ is divergence-free and tangent to the boundary, the helicity of $V$ is invariant under any volume-preserving diffeomorphism of $\Omega$ which is homotopic to the identity [3]. Our idea was to define an *iterated helicity* of $V$ by taking the helicity of $BS(V)$. Unfortunately, although $BS(V)$ is divergence-free, it is generally not tangent to the boundary, so $\mathrm{Hel}(BS(V))$ is not an invariant of $V$.

However, we can define a modified Biot-Savart operator $\widetilde{BS}(V)$ by projecting $BS(V)$ to the space of divergence-free vector fields which are tangent to the boundary of $\Omega$ and have zero flux through any cross-sectional surface. In fact, by translating into the language of differential forms, we can make sense of this "fluxless" Biot-Savart operator on any compact, oriented, smooth Riemannian 3-manifold $M$ (with or without boundary). We then define the $n$th iterated helicity of $V$ as

$$\mathrm{Hel}^n(V) := \mathrm{Hel}(\widetilde{BS}^{n-1}(V)) = \int_M \widetilde{BS}^{n-1}(V) \cdot \widetilde{BS}^{n}(V)\,\mathrm{dvol}.$$

**Theorem 3.6** (with Cantarella [11]). *If $V$ is a divergence-free vector field on a compact, oriented, Riemannian 3-manifold $M$ (with or without boundary) which is in the image of curl, the $n$th iterated helicity* $\mathrm{Hel}^n(V)$ *is invariant under volume-preserving diffeomorphisms homotopic to the identity. Moreover, the field energy $E(V)$ is bounded below by*

$$\frac{1}{\lambda_1^n}|\mathrm{Hel}^n(V)| \le |E(V)|,$$

*where $\lambda_1$ is the first positive eigenvalue of curl.*

On symmetric domains like $S^3$ or the ball $B^3$, it is straightforward to construct a vector field $V$ such that $\mathrm{Hel}^k(V) = 0$ for all $k < n$ but $\mathrm{Hel}^n(V) \neq 0$ for any positive integer $n$. Thus, our iterated helicities are higher helicities in the spirit of Arnol′d and Khesin.

Using Theorem 3.6, we find a countable basis for a space of vector fields containing all fields which can be deformed to have arbitrarily small energy (so-called "Zeldovich fields" after an example of Zeldovich that Freedman [24] proved has this property). This gives a partial characterization of Zeldovich fields and implies that there are no Zeldovich fields on certain asymmetric manifolds.

## 4. Legendrian knots and contact geometry

Another realm in which geometry and topology are closely related is that of contact geometry. A pair $(Y, \xi)$ is called a *contact manifold* if $Y$ is an oriented 3-manifold and $\xi$ is a 2-plane field on $Y$ that is completely nonintegrable. For example, the complex tangencies to $S^3 \subset \mathbb{C}^2$ form a contact structure. A *Legendrian knot* is an embedding of $S^1$ into a contact manifold such that the tangent directions are contained in the contact planes.

Legendrian contact homology is one of the most powerful invariants of Legendrian knots in $(\mathbb{R}^3, \xi_{\text{std}})$, the standard contact structure on $\mathbb{R}^3$. This invariant is a differential graded algebra $(\mathcal{A}, \partial)$ which was defined by Chekanov [12] and Eliashberg [21] as a specialized variant of symplectic field theory [22]. Ng's characteristic algebra [51] is a quotient of the Chekanov–Eliashberg DGA. Both of these invariants are highly non-commutative and difficult to compute, but they seem to capture significant geometric information about Legendrian knots.

### 4.1. Legendrian contact homology and nondestabilizability.

For example, both the Legendrian contact homology and Ng's characteristic algebra vanish for stabilized knots, but Ng conjectured that the characteristic algebra (and, thus, the Legendrian contact homology) should be nonvanishing for nondestabilizable Legendrian knots.

**Conjecture 4.1** (Ng [51])**.** *If a Legendrian link $\mathcal{K}$ has trivial characteristic algebra then it is Legendrian isotopic to a stabilization*

This conjecture is supported by the fact that both invariants are nonvanishing for every nondestabilizable Legendrian knot for which they have been computed. However, prior to my work with Vela-Vick [61], all such examples had maximal Thurston–Bennequin invariant, which, as observed by Etnyre and Honda [23], is not true of all nondestabilizable Legendrian knots.

Vela-Vick and I gave some evidence for Ng's conjecture by providing the first example of a Legendrian knot with nonvanishing characteristic algebra (and, hence, Legendrian contact homology) which does not have maximal Thurston–Bennequin invariant.

**Theorem 4.2** (with Vela-Vick [61])**.** *The Legendrian contact homology and characteristic algebra of Chongchitmate and Ng's nondestabilizable Legendrian $m(10_{161})$ are nonvanishing.*

Similar nonvanishing results hold for other examples of nondestabilizable Legendrian knots with nonmaximal Thurston–Bennequin invariant, all of which come from Chongchitmate and Ng's conjectural atlas of low-crossing Legendrian knots [14].

### 4.2. Legendrian contact homology and rulings.

A more subtle geometric property of Legendrian knots detected by Legendrian contact homology is the existence of a (graded) normal ruling. Given a *front diagram* (i.e. projection to the $xz$-plane) of a Legendrian knot in $(\mathbb{R}^3, \xi_{\text{std}})$, a *ruling* is a pairing between the left and right cusps of the diagram along with, for each pair, two paths in the diagram that join them. If these paths satisfy certain technical conditions the ruling is called a *normal ruling* or a *graded normal ruling*. The number of graded normal rulings is invariant under Legendrian isotopy [13].

Normal rulings are detected by augmentations of the Legendrian contact homology. An *augmentation* is an algebra map $\varepsilon : \mathcal{A} \to \mathbb{Z}/2$ such that $\varepsilon \circ \partial = 0$ and $\varepsilon(1) = 1$.

**Theorem 4.3** (Fuchs [26], Fuchs–Ishkanov [27], Sabloff [56])**.** *The Legendrian contact homology of a Legendrian knot $\mathcal{K}$ has a graded augmentation if and only if every front diagram of $\mathcal{K}$ has a graded normal ruling.*

The correspondence between augmentations and rulings (which is not, unfortunately, one-to-one) was clarified by Ng and Sabloff [52]. Recently Henry [31] has defined a refinement of the notion of ruling called a *Morse complex sequence* and demonstrated a more natural (and conjecturally bijective) correspondence between Morse complex sequences and augmentations.

It is straightforward to check that an augmentation factors through the abelianized characteristic algebra, and so Legendrian knots with graded normal rulings have nontrivial abelianized characteristic algebras. Sabloff, Vela-Vick, and I conjecture that the converse is also true.

**Conjecture 4.4.** *If a Legendrian knot $\mathcal{K}$ has nontrivial abelianized characteristic algebra, then any front diagram of $\mathcal{K}$ admits a graded normal ruling.*

Our approach is to try to prove the contrapositive. Specifically, given a front diagram for a Legendrian knot $\mathcal{K}$ which does not admit any rulings, we start pushing *partial rulings* across the diagram and analyze where and how the partial rulings fail to become a ruling. This is almost certainly reflected in the Legendrian contact homology, though how is not yet clear to us.

The notion of a partial ruling naturally fits into the half diagrams on which Sivek [62] defined a bordered version of Legendrian contact homology, so a first step is to show that the ruling-augmentation correspondence holds in the bordered world.

**Theorem 4.5** (with Sabloff and Vela-Vick [57])**.** *For a Legendrian knot $\mathcal{K}$ with left half $\mathcal{K}^A$ and right half $\mathcal{K}^D$, there is a surjection from the set of partial Morse complex sequences on $\mathcal{K}^A$ to the set of augmentations of the bordered contact homology $A(\mathcal{K}^A)$; likewise there is a surjection from the partial Morse complex sequences on $\mathcal{K}^D$ to the augmentations of $D(\mathcal{K}^D)$.*

This is precisely the bordered analogue of Henry's correspondence and is, like Henry's correspondence, conjecturally bijective.

## 5. Future research

### 5.1. Asymptotic behavior of Poincaré duality angles.

Theorem 2.1 and related examples suggest the following conjecture:

**Conjecture 5.1.** *Let $M^m$ be a closed, smooth, oriented Riemannian manifold and let $N^n$ be a closed submanifold of codimension $m - n \geq 2$. Define the compact Riemannian manifold*

$$M_r := M - \nu_r(N),$$

*where $\nu_r(N)$ is the open tubular neighborhood of radius $r$ about $N$. Then if $\theta_r^p$ is a Poincaré duality angle of $M_r$ in dimension $p$,*

$$\theta_r^p = O(r^{m-n})$$

*for $r$ near zero.*

This special case of the rather vague hypothesis that Poincaré duality angles measure how "close" a manifold is to being closed seems amenable to analysis. My current approach is to extend elements of $\mathcal{H}_N^p(M_r)$ and $\mathcal{H}_D^p(M_r)$ harmonically across the complement of $M_r$ inside $M$, which induces an embedding of $\mathcal{H}_N^p(M_r)$ and $\mathcal{H}_D^p(M_r)$ into the Sobolev space $H^1\Omega^p(M)$. The weaker conjecture that $\theta_r^p \to 0$ as $r \to 0$ is then equivalent to showing that $\mathcal{H}_N^p(M_r)$ and

$\mathcal{H}^p_D(M_r)$ converge weakly in $H^1$ (and thus strongly in $L^2$) to the space $\mathcal{H}^p(M)$ of harmonic fields on the closed manifold $M$.

5.2. **Reconstructing a manifold from boundary data.** Since, the "complete" Dirichlet-to-Neumann map $\Pi$ encodes at least as much information as $\Lambda$ and since the proof of Theorem 2.6 is much easier than that of Theorem 2.3, Sharafutdinov and I believe that it should be much easier to recover the full cup product structure on $M$ from the data $(\partial M, \Pi)$.

Also, the result of Krupchyk, Lassas, and Uhlmann suggests that $\Pi$ is the right operator to use when attempting to recover the Riemannian metric on $M$. One possible line of attack in the non-analytic case is: in two dimensions, Belishev [4, 6] showed that a Riemann surface can be reconstructed as the spectrum of a Banach algebra associated to the kernel of the operator $\mathrm{Id} + T^2$ (recall that $T = d\Lambda^{-1} = d\Phi^{-1}$ is the Hilbert transform). Theorem 2.4 and its proof provide significant insight into the operator $T^2$, so I am in good position to try to generalize Belishev's result to higher dimensions.

5.3. **Configuration spaces and higher-order linking invariants.** Cohen, Komendarczyk, and I believe that Theorem 3.4 can be used to express Milnor's $\bar{\mu}$-invariants in terms of Chen's iterated integrals, as first proposed by Kohno [37]. Theorem 3.4 also provides the basis for finding geometric linking integrals for higher-order linking invariants in the style of Theorem 3.2.

We are currently investigating whether Theorem 3.5 can be combined with the iterated product structure of $\mathcal{H}(n)$—perhaps in conjunction with the techniques developed to prove Theorem 3.2—to provide a full proof of Koschorke's conjecture that link-homotopy invariants of links are reflected by homotopy invariants of their associated maps.

5.4. **Higher helicities.** I plan to investigate whether asymptotic analyses of the integral formula for the triple linking number given in Theorem 3.2 or of integral formulas for other link-homotopy invariants lead to higher helicities. One intriguing possibility: although the invariant $\bar{\mu}_{1122}$—also known as the Sato-Levine invariant—is only a link-concordance invariant and not a link-homotopy invariant, it has exactly the right symmetries to provide an invariant of vector fields. I intend to work on finding a geometric integral formula for this invariant.

Theorem 3.6 is still quite new, so its consequences are largely unknown. The iterated helicities approach seems to be a genuinely novel way of defining higher helicities, and I am excited to see where this approach leads, both mathematically and in physical contexts, and to understand the consequences of Theorem 3.6.

5.5. **Legendrian knots and contact geometry.** Vela-Vick and I believe that Conjecture 4.1 is false: we have several examples of Legendrian knots with vanishing Legendrian contact homology but which do not appear to be stabilizations. We believe that studying the invariants of the doubles of these examples may lead to a proof of nondestabilizability and, thus, to a counterexample to Conjecture 4.1.

Sabloff, Vela-Vick, and I are currently investigating how Theorem 4.5 relates to Conjecture 4.4. We are also investigating whether Theorem 4.5 can be used to provide a simpler proof of the fact, originally due to Rutherford [55], that the Kauffman bound on the Thurston–Bennequin number of a Legendrian knot is sharp if and only if a front diagram for the Legendrian knot admits an (ungraded) ruling.

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