Linear-time algorithms for Monadic Logic

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Abstract

In general, the computational effort required solving a problem described by a first-order or fixed-point query (logical formula) requires time polynomial in the size of the database (finite structure). We show how a linear-time evaluation algorithm for first-order logic on bounded-degree data structures can be extended to monadic fixed-point formulas, pointing the way to a logical characterization of linear-time computability.

It is well-known that first-order queries can be evaluated in logarithmic-space on arbitrary finite structures. On bounded-degree graphs, it is surprising that first-order sentences can be evaluated in linear-time [1]. In this short presentation we illustrate how to also do this for monadic fixed-point formulas.

Let $\phi(x; S)$ be a first-order formula of one free variable in the language of graphs (vertices V with one binary edge relation E) with equality, with an additional unary relation S, appearing only positively. Write $\phi(S) = \{s : G \models \phi(s; S)\}$ where the finite graph $G = \langle V, E\rangle$ is understood to have degree at most d.

Def: The *least fixed-point* of ϕ, denoted ϕ^∞, is the smallest relation S satisfying $\phi(S) = S$.

It can be computed by any monotone method which stays below the fixed-point.

Lemma: If $S \subset \phi^\infty$ is strictly below the fixed-point, then: (a) $\exists s \in \phi(S) \bullet s \notin S$; & (b) $\forall s \in \phi(S) \bullet s \in \phi^\infty$.

In particular, a greedy method can be used in which only one element is thrown in at each step.

Corollary: If $s_i \in \phi(\{s_1,\ldots, s_{i-1}\})$ is a maximal length k sequence of distinct nodes, then $\{s_1,\ldots, s_k\} = \phi^\infty$.

Realize our graphs now include a unary relation S. Define the r-type of a node s as the isomorphism type of the neighborhood of radius r about s. Use $G_1 \cong_{r,t} G_2$ to denote two graphs that realize the same number of r-types with the same multiplicity up to a certain threshold t.

Lemma: Given $\phi(x)$, there is a radius r and a threshold t such that if $G_1 \cong_{r,t} G_2$ are of degree at most d, and s_1 and s_2 have the same r-type, then $G_1 \models \phi(s_1) \Leftrightarrow G_2 \models \phi(s_2)$.

So whether G satisfies $\phi(v)$ depends only the quantity up to t of each r-type, together with r-type of v.

Corollary: Let $\tau(v)$ be the r-type of v in G, and let $\#\tau^G = |\{v \in G: \tau(v) = \tau\}|$ for any r-type τ. Then $T(G) = \{(\tau, \min\{t, \#\tau\}) : \tau \text{ is an } r\text{-type}\}$, together with $\tau(v)$, determines whether $G \models \phi(v)$.

The algorithm assumes a unit-cost RAM model with $O(\log n)$-bit word size, in which the edge relation of the degree d graph is represented by a doubly-linked incoming and outgoing pointers at each vertex. Our method starts with $S = \varnothing$, and marks one element at a time until $S = \phi^\infty$. We always select an element in constant time that currently satisfies $\phi(S)$, keeping track of S by directly marking nodes in G, obtaining

Theorem: Over bounded-degree graphs, there is a linear-time algorithm to compute monadic fixed-points.

[1] Detlef Seese, "Linear-time computable problems and first-order descriptions" *Mathematical Structures in Computer Science*, 6(6):505-526, December 1996.