

A normal form for first-order logic over physically feasible data structures

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Logic and Databases workshop

Questions

1. What is the simplest kind of vocabulary retaining the full expressive power of first-order logic?
 2. What (infinite) classes of finite data structures are physically realizable (i.e. scale feasibly)?
- **Simplify:** find a “natural” more elegant approach which does not use actual relations in the classical sense – just predicates and functions!

Singularly logic

Vocabulary: one-place symbols (no commas allowed)

$$\mathbf{S} = \langle S, P_1, \dots, P_m, f_1, \dots, f_k \rangle$$

Monadic predicates: $P(_)$ $P \subseteq S$ *unary relation*
Monadic functions: $f(_)$ $f: S \rightarrow S$ *unary mapping*

Atomic forms: $P(F(x))$ $F = f \circ \dots$
 involving equality: $F(y) = G(z)$ [not a true relation]

Transformations

L-structure of arity k singularly *L**-structure
 $\mathbf{S} = \langle S, R_1, \dots, R_m \rangle$ $\mathbf{S}^* = \langle S^*, P_1, \dots, P_m, f_1, \dots, f_k \rangle$

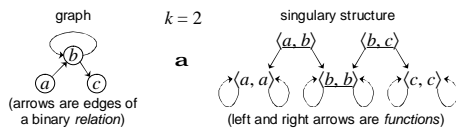
$$R_i(s_1, \dots, s_j, \dots, s_k) \left. \begin{array}{l} \text{a } k\text{-ary fact} \\ \text{where } s_j \in S \end{array} \right\} \mathbf{a} \left\{ \begin{array}{l} \langle s_1, \dots, s_j, \dots, s_k \rangle \in P_i \\ \downarrow f_j \\ \langle s_j, \dots, s_j \rangle \in S^k \end{array} \right.$$

In a sense, it preserves the *true size* of \mathbf{S} (as a database):
 $|\mathbf{S}| = |S| + |R_1| + \dots + |R_m| - |\mathbf{S}^*| = O(|S^*|)$.

Theorem: For each first-order *L*-sentence θ there is a first-order *L**-sentence θ^* such that $\mathbf{S} \models \theta$ iff $\mathbf{S}^* \models \theta^*$.

Bounded-degree classes

out-degree = k , in-degree is preserved (up to a constant)



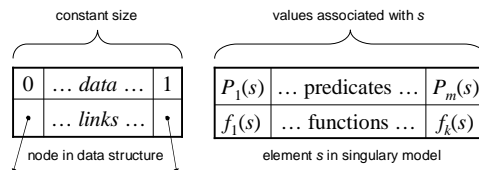
Definition: The degree of \mathbf{S} is that of its *Gaifman graph*, relating a and b iff they appear jointly in \mathbf{S} .

Theorem: A class of structures is of bounded degree iff the corresponding class of singularly structures is also.

Models for data structures

Information is stored inside nodes of fixed width.

- The number of nodes determines *size* of structure.
- The links between nodes determines its *shape*.



Physical feasibility

Assume actual implementations use resources:

- *matter* (quantized) occupies *space* at limited density
- Total amount of resources \propto size (no. of nodes).

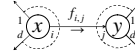
Nodes are uniform (no *overcrowding* allowed):
 $O(1)$ *bits* (mass) and $O(1)$ *connections* (space)

Unbounded in-degree is physically untenable, even though no bound on lengths of connections.

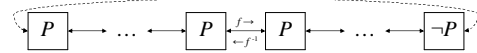
- Conclusion: feasible structures have reversible links

Reversible a bijections

Construct partial injections $f_{i,j}^{-1} = f_{j,i}$ (where defined):



Convert (finite) f -chains to cycles using *wrap-around*:



Removes *nils* – results in a *bijective* singularly model:

$$\langle D, P_1, \dots, P_m, f_1, \dots, f_k, f_1^{-1}, \dots, f_k^{-1} \rangle$$

Normal form result

Definition: For each $n \geq 1$, the *numerical quantifier* $\exists^n x \dots$ means “there are (at least) n x ’s such that ...”.

Theorem: Any formula $\theta(x_1, \dots, x_l)$ in bijective singularly logic can be rewritten as a Boolean combination of:

formulas: $\alpha(x_1, \dots, x_l)$ where α is atomic,
sentences: $\exists^n x \beta(x)$ β is quantifier-free

Proof: syntactic – the key idea is to solve equations:

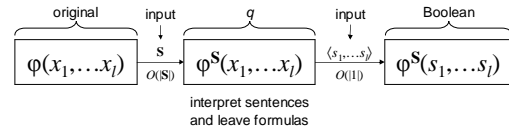
$$F(x) = G(y) \quad \mathbf{a} \quad x = F^{-1} \circ G(y)$$

Linear-time evaluations

Given: a first-order formula $\varphi(x_1, \dots, x_l)$.

Problem: to compute $\{ \langle s_1, \dots, s_l \rangle : \mathbf{S} \models \varphi(s_1, \dots, s_l) \}$.

Solution: Calculate a *partial interpretation* query $q = \varphi^S(x_1, \dots, x_l)$ using time linear in $|\mathbf{S}|$, so that $\varphi^S(s_1, \dots, s_l)$ can be computed in constant time (i.e. q demonstrates an immediate knowledge of the answer). The method:



Answers

Singularly vocabularies are the simplest kind of signature which retain the full power of FOL.

- Any structure can be mapped to a singularly one, in which size and degree are preserved uniformly.

Data structures are really just singularly models.

- Physically implementable ones are reversibly-linked, in which every function can be made bijective.

Elementary definability over asymptotically scalable classes is much simpler than it appears.

- Using numerical quantifiers, no nesting is required, and all formulas can be evaluated in linear-time.

References

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- [Hanf] ‘Model-theoretic methods in the study of elementary logic’ 1965.
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A full paper is available from my web page.