Inductive Classes of Finite Models

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In this very short paper, we bring together some very basic definitions and facts regarding the inductive definability of classes of finite structures. In particular, we give a necessary and sufficient condition on the expressibility of finiteness, the property that the domain of a structure is finite.

There is an essential distinguishing feature between elementary and inductive sentences: the ability to express finiteness. First, some background.

Definition: $\boldsymbol{K}$ is an *elementary class* of structures if there is a first-order sentence ϕ such that:

$$\mathbf{A} \models \phi \iff \mathbf{A} \in \boldsymbol{K}$$

A fundamental limitation on elementary classes is the following fact - a standard corollary of the compactness theorem.

Fact: Let $\boldsymbol{K}$ be an elementary class containing arbitrarily large finite structures. Then $\boldsymbol{K}$ must contain an infinite structure.

A first-order formula $\phi(\mathbf{x}, S)$ is *operative* if length($\mathbf{x}$) = arity(S). For an operative S-positive first-order formula $\phi(\mathbf{x}, S)$ and any ordinal ξ, let $\phi^{\xi} = \phi(\phi^{<\xi})$, where $\phi^{<\xi} = \cup\{\phi^{\zeta}: \zeta<\xi\}$ (n.b. $\phi^{<0} = \varnothing$). Let $||\phi||^{\mathbf{A}}$, the *norm* of ϕ on $\mathbf{A}$, be the least ordinal κ such that $\phi^{\kappa} = \phi^{<\kappa}$ on $\mathbf{A}$. $\phi^{\infty} = \phi^{\kappa}$ is called the *fixpoint* of ϕ, since $\phi(\phi^{\infty}) = \phi^{\infty}$. We shall always assume at least one constant in our signature, and allow substitution of constants in the fixed-point, $\phi^{\infty}(\mathbf{c}, \mathbf{y})$. The resulting class of *positive elementary inductions* is closed under all first-order operations and composition (nested recursion) [Immerman, Moschovakis].

Definition: A class of structures $\boldsymbol{K}$ is an *inductive class* if there is a first-order S-positive formula $\phi(\mathbf{x}, S)$ of arity k, and a k-tuple of constant symbols $\mathbf{c}$, such that:

$$\mathbf{A} \models \phi^{\infty}(\mathbf{c}) \iff \mathbf{A} \in \boldsymbol{K}.$$

The following theorem shows that it is possible to determine if the length of an induction is finite or infinite. This will eventually allow us to construct inductive classes which do not contain any infinite models.

Theorem: Let $\phi(\mathbf{x}, S)$ be an operative S-positive first-order formula. Determining the finitude of the closure ordinal is inductive. I.e. there is a positive elementary finite_ϕ such that:

$$\mathbf{A} \models \mathit{finite}_\phi(\boldsymbol{c}) \quad \Leftrightarrow \quad \|\phi\|^{\mathbf{A}} \text{ is finite}$$

Proof: We shall use Moschovakis' notation. In particular, let $|\mathbf{x}|$ be the least ordinal λ such that $\phi^\lambda(\mathbf{x})$ holds. In this case, $\mathbf{x}$ is said to appear in the λ^{th} stage of ϕ. If $\mathbf{x}$ is outside of ϕ^∞, then we set $|\mathbf{x}|$ to be ∞, which is interpreted as being larger than any ordinal. I.e. for any $\mathbf{x}$, $|\mathbf{x}| < \infty \Leftrightarrow \phi^\infty(\mathbf{x})$. Note that $0 \le |\mathbf{x}| < \|\phi\|$. Moschovakis has shown that the following inequality relations on stages are inductive [Stage Comparison Theorem].

$$|\mathbf{x}| \le |\mathbf{y}| \;\&\; |\mathbf{x}| < \infty$$
$$|\mathbf{x}| < |\mathbf{y}|$$

From these, we can express the property that $|\mathbf{y}|$ is a successor ordinal: [Lindell]

$$\mathit{succ}(\mathbf{y}) \quad \Leftrightarrow \quad (\exists \mathbf{z})[|\mathbf{z}| < |\mathbf{y}| \wedge \phi(\mathbf{y}, \{\mathbf{u} : |\mathbf{u}| \le |\mathbf{z}|\})] \qquad \text{i.e. } |\mathbf{y}| = \zeta + 1 \text{ for some } \zeta.$$

Now see that the comparison of non-adjacent stages is inductive: [Immerman]

$$\mathbf{x} \ll \mathbf{y} \quad \Leftrightarrow \quad |\mathbf{x}| \le |\mathbf{x}| \wedge \neg\phi(\mathbf{y}, \{\mathbf{u} : \neg(|\mathbf{x}| < |\mathbf{u}|)\})] \qquad \text{i.e. } |\mathbf{x}| + 1 < |\mathbf{y}|$$

Then we can say that $\mathbf{y}$ is in a maximal stage of ϕ, (which does not exist if $\|\phi\|$ is a limit ordinal or zero). We can express this property as: [Immerman]

$$\mathit{max}(\mathbf{y}) \quad \Leftrightarrow \quad (\forall \mathbf{z})[(|\mathbf{z}| \le |\mathbf{y}|) \vee (|\mathbf{y}| \ll |\mathbf{z}|)] \qquad \text{i.e. } |\mathbf{y}| = \|\phi\| - 1.$$

Note that $\mathit{max}(\mathbf{y}) \Rightarrow \phi^\infty(\mathbf{y})$, since $\neg\phi^\infty(\mathbf{y}) \Rightarrow \neg(|\mathbf{y}| \le |\mathbf{y}|) \wedge \neg(|\mathbf{y}| \ll |\mathbf{y}|)$.

If ϕ has a maximal stage, we can say that $\mathbf{y}$ is outside the fixed-point of ϕ. [Immerman]

$$\mathit{not}(\mathbf{y}) \quad \Leftrightarrow \quad (\exists \mathbf{z})[\mathit{max}(\mathbf{z}) \wedge (|\mathbf{z}| < |\mathbf{y}|)] \qquad \text{i.e. } |\mathbf{y}| > \|\phi\| - 1.$$

Now, $\|\phi\|$ is finite if and only if $\|\phi\| = 0$, or ϕ has a maximal stage and every stage of ϕ except the first has a preceding stage. N.b. the original mistake in [Lindell] which forgot the first stage, $\phi^0(\mathbf{x})$, and the case $\|\phi\| = 0$.

$$\mathit{finite}_\phi \quad \Leftrightarrow \quad \neg(\exists \mathbf{x})\phi^0(\mathbf{x}) \vee (\exists \mathbf{y})\mathit{max}(\mathbf{y}) \wedge (\forall \mathbf{x})[\mathit{not}(\mathbf{x}) \vee \mathit{succ}(\mathbf{x}) \vee \phi^0(\mathbf{x})]$$

By the closure properties of the class of inductive queries [Moschovakis], finite_ϕ is inductive. QED

Definition: $\boldsymbol{K} = \{\mathbf{A} : \mathbf{A} \models T, \varphi^\infty(\mathbf{c})\}$ is an *inductive extended elementary class* if T is a set of first-order sentences, and $\varphi^\infty(\mathbf{c})$ is an inductive sentence for a tuple of constants $\mathbf{c}$.

Note: It is sufficient to consider a single simple fixed-point instead of a finite set of operative systems because of the closure properties of positive elementary inductions [Moschovakis].

Definition: Let θ be an S-positive first-order formula, and let $\boldsymbol{K}$ be a class of arbitrarily large finite structures. Then θ is called an *increasing induction* if $\lim \{\|\theta\|_{\mathbf{A}} : \mathbf{A}\in\boldsymbol{K}\} = \infty$. Note the distinction between unbounded inductions $\sup \{\|\theta\|_{\mathbf{A}} : \mathbf{A}\in\boldsymbol{K}\} = \infty$ [McColm] versus increasing inductions (where the limit must exist).

Theorem: Let $\boldsymbol{K}$ be a class of arbitrarily large finite structures. Then $\boldsymbol{K}$ is an inductive extended elementary class iff $\boldsymbol{K}$ admits an increasing induction θ.

Note: $\boldsymbol{K}$ is always the *finite part* of an extended elementary class by just stating which finite structures are not in $\boldsymbol{K}$. It is similarly just a subset of an inductive class of finite structures.

Proof: ($\Rightarrow$) Let $\boldsymbol{K} = \{\mathbf{A} : \mathbf{A} \models T, \varphi^{\infty}(\mathbf{c})\}$, and suppose towards a contradiction that there is an infinite $\boldsymbol{K}' \subseteq \boldsymbol{K}$ such that $\|\varphi\|^{\boldsymbol{K}'}$ is bounded by m. Then $\boldsymbol{K}' \models (\forall\mathbf{x})[\varphi^{m}(\mathbf{x}) \leftrightarrow \varphi^{m-1}(\mathbf{x})]$, and hence $T, (\forall\mathbf{x})[\varphi^{m}(\mathbf{x}) \leftrightarrow \varphi^{m-1}(\mathbf{x})], \varphi^{m}(\mathbf{c})$ is a first-order theory with arbitrarily large finite models (namely those in $\boldsymbol{K}'$). Therefore, by compactness, there is a infinite model $\mathbf{A}^*$ of this theory, which implies that $\mathbf{A}^* \models T, \varphi^{\infty}(\mathbf{c})$, and hence $\mathbf{A}^* \in \boldsymbol{K}$, a contradiction.
($\Leftarrow$) Let $T = \mathrm{Th}(\boldsymbol{K})$, and $\varphi^{\infty}(\mathbf{c}) \equiv \mathrm{Finite}_{\theta}$. Clearly, $\boldsymbol{K} \models T, \varphi^{\infty}(\mathbf{c})$. If we are given any finite $\mathbf{A} \notin \boldsymbol{K}$, then there is a specific sentence of $\mathrm{Th}(\boldsymbol{K})$ which excludes $\mathbf{A}$. Now, by hypothesis, for every m there is an n such that $(\exists x_1\ldots x_n) \wedge_{i\ {}^{1}\ j} [x_i \neq x_j] \rightarrow (\exists\mathbf{x})[\neg\theta^{m-1}(\mathbf{x}) \wedge \theta^{m}(\mathbf{x})] \in T$. Hence, any infinite $\mathbf{A} \models \{(\exists\mathbf{x})[\neg\theta^{m-1}(\mathbf{x}) \wedge \theta^{m}(\mathbf{x})]\}$for $m = 1\ldots\infty$, which means $\|\theta\|^{\mathbf{A}} = \omega$, and hence $\mathbf{A} \models \mathrm{Finite}_{\theta}$. QED

Corollary: Let $\boldsymbol{K}$ be an inductive class of arbitrarily large finite structures. Then $\boldsymbol{K}$ admits an increasing induction. Note: $\boldsymbol{K}$ is hence proficient [see McColm for a definition].

References:

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