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Linear Order**

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# First Order Logic, Fixed Point Logic and Linear Order

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**Abstract.** The Ordered conjecture of Kolaitis and Vardi asks whether fixed-point logic differs from first-order logic on every infinite class of finite ordered structures. In this paper, we develop the tool of bounded variable element types, and illustrate its application to this and the original conjectures of McColm, which arose from the study of inductive definability and infinitary logic on proficient classes of finite structures (those admitting an unbounded induction). In particular, for a class of finite structures, we introduce a compactness notion which yields a new proof of a ramified version of McColm's second conjecture. Furthermore, we show a connection between a model-theoretic preservation property and the Ordered Conjecture, allowing us to prove it for classes of strings (colored orderings). We also elaborate on complexity-theoretic implications of this line of research.

## 1 Introduction

The extensions of first order logic by means of fixed point operators, in particular the least fixed point and partial fixed point operators, have been much studied in recent years in the field of finite model theory. This is in large measure due to their connection with complexity classes. Immerman [Imm86] and Vardi [Var82] showed that the logic LFP, the extension of first order logic with a least fixed point operator, captures the class PTIME on ordered structures. Vardi [Var82] and Abiteboul and Vianu [AV91] showed that the similar extension of first order logic with a partial fixed point operator PFP captures the class PSPACE on ordered structures. Furthermore, Abiteboul and Vianu [AV95] showed that  $LFP = PFP$  if, and only if,  $PTIME = PSPACE$ , even without the restriction to ordered structures. One of the most important tools in the analysis of the fixed point logics is the bounded variable infinitary logic  $L_{\infty\omega}^w$ . Kolaitis and Vardi

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[KV92b] showed that, on the class of finite structures, LFP and PFP can be seen as fragments of  $L_{\infty\omega}^\omega$ . Moreover,  $L_{\infty\omega}^\omega$  has an elegant characterization in terms of pebble games which has proved an extremely useful tool in the analysis of the expressive power of the fixed point logics.

The logics LFP and PFP are both extensions of first order logic, and indeed, they are proper extensions on the class of all finite structures and on the class of ordered finite structures. It also follows from the result of Abiteboul and Vianu that if we can separate these two logics on any class of finite structures  $\mathcal{C}$ , then we would separate PTIME from PSPACE. On the other hand, one can construct infinite classes of structures on which the logics are equivalent and both of them, indeed even  $L_{\infty\omega}^\omega$ , collapse to first order logic.

Kolaitis and Vardi [KV92a] initiated an investigation of which classes of structures  $\mathcal{C}$  have the property that LFP and  $L_{\infty\omega}^\omega$  collapse to first order logic on  $\mathcal{C}$ . They proved a conjecture of McColm [McC90], showing that  $L_{\infty\omega}^\omega$  collapses to FO if, and only if, every positive, first-order induction is bounded. Gurevich, Immerman and Shelah [GIS94] refuted another conjecture due to McColm by constructing a class of structures on which LFP collapses to FO, but  $L_{\infty\omega}^\omega$  does not. Kolaitis and Vardi [KV92b] conjectured the following weaker version of McColm's conjecture, which remains open:

**Conjecture 1 (Kolaitis-Vardi)** *On every infinite class of ordered structures, there is a polynomial time computable query that is not first order definable.*

In this paper, we discuss McColm's conjectures, relating them to finite variable element types as introduced in [DLW95], a notion of compactness for classes of finite structures and a preservation property. In particular, we relate this preservation property to Conjecture 1, allowing us to prove it for classes of strings (linear orders with unary relations). We also comment on the complexity theoretic implications of Conjecture 1. Parts of the material in this paper appeared in preliminary form in [Daw93].

Section 2 covers the background material on fixed point logics, infinitary logics and element types. Section 3 relates inductive definitions and McColm's conjectures to bounded variable element types, compactness and preservation properties. Section 4 discusses the relation between the preservation properties and Conjecture 1, while Section 5 relates this conjecture to questions in complexity theory.

## 2 Background

We assume the standard definitions of a first order language (or signature) and a structure interpreting it. Unless otherwise mentioned, all structures we will be dealing with are assumed to have finite universe and all signatures are assumed to be finite and relational, that is, to consist of finitely many relation symbols. We write  $\mathcal{F}_\sigma$  to denote the class of all finite structures of signature  $\sigma$ , and  $\mathcal{O}_\sigma$  to denote the class of ordered finite  $\sigma$ -structures, i.e.  $\mathcal{O}_\sigma$  is the collection of

structures in  $\mathcal{F}_{\sigma \cup \{\leq\}}$  which interpret the binary relation symbol  $\leq$  as a linear order.

An  $n$ -ary query over a class of structures  $\mathcal{C}$  is a map  $Q$  sending each structure  $\mathfrak{A} \in \mathcal{C}$  to an  $n$ -ary relation over  $\mathfrak{A}$  which satisfies the following condition: for all  $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$ , if  $f$  is an isomorphism from  $\mathfrak{A}$  onto  $\mathfrak{B}$ , then  $Q(\mathfrak{B}) = f[Q(\mathfrak{A})]$ .

We will write FO, LFP, *etc.* both to denote logics (i.e. sets of formulas) and the classes of queries that are expressible in the respective logics. We say a logic  $L$  collapses to another logic  $L'$  over a class of structures  $\mathcal{C}$ , if and only if, the collection of restrictions of queries in  $L$  to  $\mathcal{C}$  is included in the collection of restrictions of queries in  $L'$  to  $\mathcal{C}$ .

## 2.1 Inductive and Infinitary Logics

Let  $\varphi(R, x_1, \dots, x_k)$  be a first-order formula. On a structure,  $\mathfrak{A}$ ,  $\varphi$  defines the operator,  $\Phi_{\mathfrak{A}}(R^{\mathfrak{A}}) = \{\langle a_1, \dots, a_k \rangle \mid \langle \mathfrak{A}, R^{\mathfrak{A}} \rangle \models \varphi[a_1, \dots, a_k]\}$ . If  $\varphi$  is an  $R$ -positive formula,  $\Phi_{\mathfrak{A}}$  is monotone. We may view  $\varphi$  as determining an induction on  $\mathfrak{A}$  the stages of which are defined as follows:  $\varphi_{\mathfrak{A}}^0 = \emptyset$ ;  $\varphi_{\mathfrak{A}}^{m+1} = \Phi_{\mathfrak{A}}(\varphi_{\mathfrak{A}}^m)$ . The *closure ordinal* of  $\varphi$  on  $\mathfrak{A}$ , denoted  $\|\varphi\|_{\mathfrak{A}}$ , is the least  $m$  such that  $\varphi_{\mathfrak{A}}^m = \varphi_{\mathfrak{A}}^{m+1}$ . The  $m^{\text{th}}$  stage of the induction determined by  $\varphi$  can be uniformly defined over all structures by a first-order formula which we denote by  $\varphi^m$ . The set inductively defined by  $\varphi$  on  $\mathfrak{A}$ , denoted  $\varphi_{\mathfrak{A}}^{\infty}$ , is the least fixed point of the operator  $\Phi_{\mathfrak{A}}$ , that is,  $\varphi_{\mathfrak{A}}^{\infty} = \varphi_{\mathfrak{A}}^m$ , where  $m = \|\varphi\|_{\mathfrak{A}}$ . If  $s$  is a  $k$ -tuple of elements of  $\mathfrak{A}$  and  $s \in \varphi_{\mathfrak{A}}^{\infty}$  we use  $|s|_{\varphi}$  to denote the least  $m$  such that  $s \in \varphi_{\mathfrak{A}}^m$ . The *stage comparison query* for  $\varphi$ , denoted  $\leq_{\varphi}$ , is the query which assigns to each structure  $\mathfrak{A}$  the  $2k$ -ary relation defined as follows:

$$s \leq_{\varphi} s' \iff s \in \varphi_{\mathfrak{A}}^{\infty} \wedge s' \in \varphi_{\mathfrak{A}}^{\infty} \wedge |s|_{\varphi} \leq |s'|_{\varphi}$$

where  $s$  and  $s'$  are  $k$ -tuples of elements of  $\mathfrak{A}$ .

We write LFP for the extension of first-order logic with the **lfp** operation which uniformly determines the least fixed point of an  $R$ -positive formula. That is, for any  $R$ -positive formula  $\varphi$ ,  $\mathbf{lfp}(R, x_1, \dots, x_k)\varphi$  is a formula of LFP and  $\mathfrak{A} \models \mathbf{lfp}(R, x_1, \dots, x_k)\varphi[s]$ , if and only if,  $s \in \varphi_{\mathfrak{A}}^{\infty}$ . We will need the following basic result about inductive definability which is a special case of Moschovakis's Stage Comparison Theorem (see [Mos74]).

**Theorem 2 [Mos74].** *Let  $\varphi$  be an  $R$ -positive first-order formula. The stage comparison query  $\leq_{\varphi}$  is definable in LFP.*

The stages  $\varphi_{\mathfrak{A}}^m$  can be defined for an arbitrary (not necessarily positive) formula  $\varphi$  on a structure  $\mathfrak{A}$ . If the formula is not positive, these stages are not necessarily increasing, and they may or may not converge to a fixed point. We define the partial fixed point of  $\varphi$  on structure  $\mathfrak{A}$  to be  $\varphi_{\mathfrak{A}}^m$ , for  $m$  such that  $\varphi_{\mathfrak{A}}^m = \varphi_{\mathfrak{A}}^{m+1}$ , if such an  $m$  exists and empty otherwise. The logic PFP is then the closure of first order logic under an operation **ppf** uniformly defining the partial fixed point of a formula.

The interest in fixed point logics on finite structures stems largely from their connection with complexity classes, as established by the following results.

**Theorem 3 [Imm86, Var82].** For any signature  $\sigma$ ,  $\text{LFP} = \text{PTIME}$  on  $\mathcal{O}_\sigma$ .

**Theorem 4 [Var82, AV91].** For any signature  $\sigma$ ,  $\text{PFP} = \text{PSPACE}$  on  $\mathcal{O}_\sigma$ .

**Theorem 5 [AV95].**  $\text{LFP} = \text{PFP}$  if and only if  $\text{PTIME} = \text{PSPACE}$ .

In particular, it follows from Theorem 5 that the separation of LFP and PFP on any class  $\mathcal{C}$  of finite structures would yield the separation of PTIME from PSPACE.

Let  $L^k$  be the fragment of first-order logic which consists of those formulas whose variables, both free and bound, are among  $x_1, \dots, x_k$ . Let  $L_{\infty\omega}^k$  be the closure of  $L^k$  under the first order operations and the operations of conjunction and disjunction applied to arbitrary (finite or infinite) sets of formulas.  $L_{\infty\omega}^\omega = \bigcup_{k \in \omega} L_{\infty\omega}^k$ . Kolaitis and Vardi [KV92b] established that on  $\mathcal{F}_\sigma$ , the fixed point logics LFP and PFP can be viewed as fragments of  $L_{\infty\omega}^\omega$ . Indeed, they establish the following result concerning the stages of a first-order induction.

**Theorem 6 [KV92b].** Let  $\varphi \in L^k$  and let  $\varphi^m$  be the  $m$ -th stage of the induction determined by  $\varphi$ .  $\varphi^m$  is uniformly definable in  $L^{2k}$  over the class of finite structures. Hence, the least fixed point of  $\varphi$  is uniformly definable in  $L_{\infty\omega}^{2k}$  over the class of finite structures.

The following definition was introduced by McColm [McC90].

**Definition 7.** A class  $\mathcal{C}$  of structures is proficient, if there is some positive formula  $\varphi$  such that  $\sup(\{|\varphi|_{\mathfrak{A}} \mid \mathfrak{A} \in \mathcal{C}\}) \geq \omega$ .

McColm [McC90] formulated two conjectures, which taken together state that the following three conditions are equivalent for any class of structures  $\mathcal{C}$ .

1.  $\mathcal{C}$  is not proficient;
2. LFP collapses to first order logic on  $\mathcal{C}$ ;
3.  $L_{\infty\omega}^\omega$  collapses to first order logic on  $\mathcal{C}$ .

It is easily seen that condition (1) implies (2), for if  $\varphi$  is a formula such that  $\sup(\{|\varphi|_{\mathfrak{A}} \mid \mathfrak{A} \in \mathcal{C}\}) < \omega$ , then there is an  $m \in \omega$  such that  $\varphi_{\mathfrak{A}}^\infty = \varphi_{\mathfrak{A}}^m$  for all  $\mathfrak{A} \in \mathcal{C}$ . But, by Theorem 6 it follows that  $\varphi^m$  is uniformly defined by a first order formula. McColm [McC90] also showed that condition (3) implies (1). Kolaitis and Vardi [KV92a] showed that (1) implies (3), thereby establishing the equivalence of (1) and (3) and resolving the second of McColm's two conjectures. Gurevich *et al.* [GIS94] construct an example of a class of structures where (2) holds but (1) fails, refuting the first of the two conjectures.

While McColm's first conjecture has been refuted in the general case, it remains open whether it nonetheless holds on classes of ordered structures, i.e. for any class  $\mathcal{C}$  that is a subclass of  $\mathcal{O}_\sigma$  for some  $\sigma$ . It was conjectured by Kolaitis and Vardi [KV92a] that it does. Since the only implication that is unresolved is the implication (2)  $\Rightarrow$  (1), this conjecture is the one stated as Conjecture 1 above.

## 2.2 Element Types

The following definition introduces the notion of element type which plays a fundamental role in our investigations.

**Definition 8.** Let  $\mathfrak{A}$  be a structure and let  $l \leq k$  be natural numbers. For any sequence  $s = \langle a_1, \dots, a_l \rangle$  of elements of  $\mathfrak{A}$ , the  $L^k$ -type of  $s$  in  $\mathfrak{A}$ , denoted  $\text{Type}_k(\mathfrak{A}, s)$ , is the set of formulas,  $\varphi \in L^k$  with free variables among  $x_1, \dots, x_l$ , such that  $\mathfrak{A} \models \varphi[a_1 \dots a_l]$ .  $\tau$  is an  $L^k$ -type, if and only if, it is the  $L^k$ -type of some tuple in some (finite or infinite) structure. If  $\tau$  is an  $L^k$ -type we say that the tuple  $s$  realizes  $\tau$  in  $\mathfrak{A}$ , if and only if,  $\tau = \text{Type}_k(\mathfrak{A}, s)$ .

In [DLW95] we established some properties of  $L^k$ -types realized in finite structures, among them the following basic result that the  $L^k$ -type of a tuple in a finite structure is determined by a single formula of  $L^k$ .

**Theorem 9 [DLW95].** *For every finite structure  $\mathfrak{A}$ , for every  $l \leq k$  and  $l$ -tuple  $s$  of elements from  $\mathfrak{A}$ , there is a formula  $\varphi \in \text{Type}_k(\mathfrak{A}, s)$  such that for any structure  $\mathfrak{B}$ , and  $l$ -tuple  $t$  of elements  $\mathfrak{B}$ , if  $\mathfrak{B} \models \varphi[t]$ , then  $\text{Type}_k(\mathfrak{A}, s) = \text{Type}_k(\mathfrak{B}, t)$ .*

If  $\varphi$  satisfies the conditions of Theorem 9 we say that  $\varphi$  isolates  $\text{Type}_k(\mathfrak{A}, s)$ .

We write  $\langle \mathfrak{A}, s \rangle \equiv^k \langle \mathfrak{B}, t \rangle$  to denote that  $\text{Type}_k(\mathfrak{A}, s) = \text{Type}_k(\mathfrak{B}, t)$ . Recall that the quantifier rank of a formula is the maximum depth of nesting of quantifiers in the formula. We write  $\langle \mathfrak{A}, s \rangle \equiv^{k,n} \langle \mathfrak{B}, t \rangle$  to denote that  $\text{Type}_k(\mathfrak{A}, s)$  and  $\text{Type}_k(\mathfrak{B}, t)$  agree on all formulas of  $L^k$  of quantifier rank  $\leq n$ . Finally, we write  $\langle \mathfrak{A}, s \rangle \equiv_{\infty\omega}^k \langle \mathfrak{B}, t \rangle$  to denote that for every formula  $\varphi \in L_{\infty\omega}^k$ ,  $\mathfrak{A} \models \varphi[s]$  if and only if  $\mathfrak{B} \models \varphi[t]$ .

Notice that by Theorem 9, for every structure  $\mathfrak{A}$  and every tuple  $s$  of elements of  $\mathfrak{A}$  of length  $\leq k$ , there is an  $n$  such that for every tuple of elements  $s'$  of  $\mathfrak{A}$ , if  $\langle \mathfrak{A}, s \rangle \equiv^{k,n} \langle \mathfrak{A}, s' \rangle$ , then  $\langle \mathfrak{A}, s \rangle \equiv^k \langle \mathfrak{A}, s' \rangle$ . This observation justifies the following definition.

**Definition 10.** Let  $\mathfrak{A}$  be a structure and  $s$  be a tuple of elements of  $\mathfrak{A}$  of length  $\leq k$ . The *Scott rank* of  $s$  in  $\mathfrak{A}$  with respect to  $k$ , denoted  $\text{sr}_{\mathfrak{A}}^k(s)$  is equal to the least  $n$  such that for every tuple of elements  $s'$  of  $\mathfrak{A}$ , if  $\langle \mathfrak{A}, s \rangle \equiv^{k,n} \langle \mathfrak{A}, s' \rangle$ , then  $\langle \mathfrak{A}, s \rangle \equiv^k \langle \mathfrak{A}, s' \rangle$ . The *Scott rank* of a structure  $\mathfrak{A}$  with respect to  $k$ , denoted  $\text{sr}^k(\mathfrak{A})$ , is equal to  $\sup(\{\text{sr}_{\mathfrak{A}}^k(s) \mid s \in |\mathfrak{A}|^{\leq k}\})$ .

We will make use of Scott ranks in obtaining information about the expressive power of LFP over arbitrary classes of finite structures. The next lemma codifies a simple relation between the Scott rank of a structure  $\mathfrak{A}$  and the number of  $L^k$ -types of  $k$ -tuples realized over  $\mathfrak{A}$ . The definition which precedes it introduces notation which will be useful here and below.

**Definition 11.** Let  $\mathfrak{A}$  be a structure, let  $\mathcal{C}$  be a class of structures, and let  $l, k$  be natural numbers with  $l \leq k$ .

1.  $S_i^k(\mathfrak{A}) = \{\text{Type}_k(\mathfrak{A}, \langle a_1, \dots, a_i \rangle) \mid a_1, \dots, a_i \in |\mathfrak{A}|\}$ .
2.  $\nu_k(\mathfrak{A}) = \text{card}(S_k^k(\mathfrak{A}))$ .
3.  $S_i^k(\mathcal{C}) = \bigcup_{\mathfrak{A} \in \mathcal{C}} S_i^k(\mathfrak{A})$ .

**Lemma 12.** *For all finite structures  $\mathfrak{A}$  and  $k \in \omega$ ,*

$$\text{sr}^k(\mathfrak{A}) \leq \nu_k(\mathfrak{A}) - 1.$$

**Proof:** Note that for each  $\mathfrak{A}$ ,  $k$ , and  $n$ ,  $\equiv^{k,n}$  and  $\equiv^k$  determine equivalence relations on the set of  $k$ -tuples of elements of  $\mathfrak{A}$ . The collection of equivalence classes determined by  $\equiv^k$  corresponds exactly to  $S_k^k(\mathfrak{A})$  and thus the number of equivalence classes is  $\nu_k(\mathfrak{A})$ . For each  $n$ , the equivalence relation  $\equiv^{k,n+1}$  is a refinement of  $\equiv^{k,n}$ . Moreover, if  $m = \text{sr}^k(\mathfrak{A})$ , then the equivalence relation  $\equiv^{k,m}$  is identical to  $\equiv^k$ . The result now follows immediately. ■

The equivalence relations  $\equiv^{k,n}$  (and consequently,  $\equiv^k$ ) can be characterized in terms of the following two-player  $k$ -pebble game. We have a board consisting of one copy of each of the structures  $\mathfrak{A}$  and  $\mathfrak{B}$ . There is also a supply of pairs of pebbles  $\{\langle a_1, b_1 \rangle, \dots, \langle a_k, b_k \rangle\}$ . At each move of the game, Player I picks up one of the pebbles (either an unused pebble, or one that is already on the board) and places it on an element of the corresponding structure (i.e. she places  $a_i$  on an element of  $A$  or  $b_i$  on an element of  $B$ ). Player II then responds by placing the unused pebble in the pair on an element of the other structure. Player II loses if the resulting map,  $f$ , from  $\mathfrak{A}$  to  $\mathfrak{B}$ , given by  $f(a_j) = b_j$ ,  $1 \leq j \leq k$ , is not a partial isomorphism. Player II wins the  $n$ -move game if she has a strategy to avoid losing in the first  $n$  moves, regardless of what moves are made by Player I. Moreover, some of the pebbles may be placed on the board before the start of the game. That is, if  $s$  is an  $l$ -tuple of elements of  $\mathfrak{A}$  and  $t$  is an  $l$ -tuple of elements of  $\mathfrak{B}$ , where  $l \leq k$ , then we say the pebbles are initially placed on  $s$  and  $t$  if before the start of the game, the pebbles  $a_1, \dots, a_l$  are on the elements of  $s$  and the pebbles  $b_1, \dots, b_l$  are on the elements of  $t$ . We then have the following characterization:

**Theorem 13 [Imm82, Poi82].** *Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be structures over a fixed signature and let  $s$  and  $t$  be tuples of elements from the respective structures. Player II wins the  $n$  move,  $k$ -pebble game on structures  $\mathfrak{A}$  and  $\mathfrak{B}$  with the pebbles initially on the tuples  $s$  and  $t$ , if and only if,  $\langle \mathfrak{A}, s \rangle \equiv^{k,n} \langle \mathfrak{B}, t \rangle$ .*

Kolaitis and Vardi [KV92b] proved that the equivalence relations  $\equiv^k$  and  $\equiv_{\infty\omega}^k$  coincide when restricted to finite structures.

**Theorem 14 [KV92b].** *For finite structures  $\mathfrak{A}$  and  $\mathfrak{B}$ , and tuples  $s$  and  $t$  of elements from the respective structures, the following are equivalent:*

- $\langle \mathfrak{A}, s \rangle \equiv^k \langle \mathfrak{B}, t \rangle$ ;
- $\langle \mathfrak{A}, s \rangle \equiv_{\infty\omega}^k \langle \mathfrak{B}, t \rangle$ .

The next result characterizes the descriptive complexity of  $L^k$ -type equivalence and establishes a further connection between finite variable element types and inductive definability.

**Theorem 15 [DLW95].** *Let  $l \leq k$  and let  $\sigma$  be a finite, relational signature. There is an  $R$ -positive first-order formula  $\zeta$  such that for any structure  $\mathfrak{A}$  of signature  $\sigma$  and any  $l$ -tuples  $s$  and  $s'$ ,  $\mathfrak{A} \models \mathbf{lfp}(R, x_1, \dots, x_{2l})\zeta[s, s']$ , if and only if,  $s$  and  $s'$  realize distinct  $L^k$ -types in  $\mathfrak{A}$ .*

In sketching a proof of this theorem, we will use the following notion of basic type.

**Definition 16.** For any structure  $\mathfrak{A}$  and elements  $a_1, \dots, a_l \in |\mathfrak{A}|$ , where  $l \leq k$  the *basic  $L^k$ -type* of  $a_1, \dots, a_l$  is the set of *atomic* formulas,  $\varphi$ , of  $L^k$  in  $l$  free variables such that  $\mathfrak{A} \models \varphi[a_1, \dots, a_l]$ .

Note that for a given finite, relational signature,  $\sigma$ , there are only finitely many distinct basic types. Furthermore, each basic type is characterized by a single quantifier free formula of  $L^k$ .

**Proof of Theorem 15 (Sketch):** Let  $\alpha_1(x_1, \dots, x_k), \dots, \alpha_q(x_1, \dots, x_k)$  be a fixed enumeration of quantifier free formulas of  $L^k$  in  $k$  free variables characterizing all the basic types in the signature  $\sigma$ . Then, define  $\varphi_0$  as follows:

$$\varphi_0(x_1, \dots, x_k, y_1, \dots, y_k) \equiv \bigvee_{1 \leq i \neq j \leq q} (\alpha_i(\bar{x}) \wedge \alpha_j(\bar{y}))$$

where  $\alpha_i(\bar{y})$  is obtained from  $\alpha_i(\bar{x})$  by replacing every  $x_j$  by  $y_j$ . It should be clear that for any tuples  $\bar{a}, \bar{b} \in |\mathfrak{A}|^k$ ,  $\mathfrak{A} \models \varphi_0[\bar{a}\bar{b}]$  if and only if the basic types of  $\bar{a}$  and  $\bar{b}$  are different.

Now, define  $\zeta$  as follows:

$$\begin{aligned} \zeta(R, x_1, \dots, x_k, y_1, \dots, y_k) \equiv & \varphi_0(\bar{x}\bar{y}) \vee \bigvee_{1 \leq i \leq k} \exists x_i \forall y_i R(x_1, \dots, x_k, y_1, \dots, y_k) \\ & \vee \bigvee_{1 \leq i \leq k} \exists y_i \forall x_i R(x_1, \dots, x_k, y_1, \dots, y_k) \end{aligned}$$

A  $k$ -pebble game argument can now be used to show that the least fixed point of  $\zeta$  expresses the *inequivalence* of  $L^k$ -types. Indeed, the  $n + 1$ -th stage of the induction determined by  $\zeta$  expresses the inequivalence of  $L^k$ -types restricted to formulas of quantifier rank at most  $n$ .  $\blacksquare$

The following lemma is a corollary to the proof of the preceding theorem. It relates Scott ranks to the stages of the induction generated by the formula  $\zeta$  in our proof sketch above.

**Lemma 17.** *Let  $l \leq k$  and let  $\sigma$  be a finite, relational signature. Let  $\zeta$  be the formula constructed above relative to  $k$  and  $\sigma$ . Let  $\mathfrak{A}$  be a structure of signature  $\sigma$ , and let  $s$  be an  $l$ -tuple of elements of  $\mathfrak{A}$  with  $\text{sr}_{\mathfrak{A}}^k(s) = m$ . Then,*

1. *there is an  $l$ -tuple  $s'$ , such that  $\mathfrak{A} \not\models \zeta^m[s, s']$  and  $\mathfrak{A} \models \zeta^{m+1}[s, s']$ .*
2. *for every  $l$ -tuple  $s'$ , if  $\mathfrak{A} \models \mathbf{lfp}(R, x_1, \dots, x_k, y_1, \dots, y_k)\zeta[s, s']$ , then  $\mathfrak{A} \models \zeta^{m+1}[s, s']$ .*

*In consequence,  $\|\zeta\|_{\mathfrak{A}} = \text{sr}^k(\mathfrak{A}) + 1$ .*

Moreover, a stronger form of Theorem 15 can be shown, namely that the equivalence classes with respect to  $L^k$  in a structure  $\mathfrak{A}$  can, in some sense, be ordered uniformly by a formula of LFP. This result, stated formally below, is a crucial step in the proof of the result due to Abiteboul and Vianu that  $LFP = PFP$  if and only if  $P=PSPACE$ .

**Theorem 18 [AV95, DLW95].** *For every  $k$  and any signature  $\sigma$ , there is an LFP formula  $\xi(x_1, \dots, x_{2k})$  such that for any  $\sigma$ -structure  $\mathfrak{A}$  and  $k$ -tuples  $s, s'$  and  $s''$  of elements of  $\mathfrak{A}$ :*

1. if  $\mathfrak{A} \models \xi[s, s']$  then  $s$  and  $s'$  realize distinct  $L^k$  types in  $\mathfrak{A}$ ;
2. it is not the case that  $\mathfrak{A} \models \xi[s, s']$  and  $\mathfrak{A} \models \xi[s', s]$ ; and
3. if  $s$  and  $s'$  realize distinct  $L^k$  types in  $\mathfrak{A}$  then either  $\mathfrak{A} \models \xi[s, s']$  or  $\mathfrak{A} \models \xi[s', s]$ .
4. if  $\mathfrak{A} \models \xi[s, s']$  and  $\mathfrak{A} \models \xi[s', s'']$  then  $\mathfrak{A} \models \xi[s, s'']$ .

We will use the symbol  $<_k$  to denote the pre-order on  $k$ -tuples defined by the formula  $\xi$ .

### 3 Element Types and Inductive Definitions

In this section, we use the machinery of  $L^k$ -types developed above to provide a proof of McColm's second conjecture and related results. The definition of proficiency of a class  $\mathcal{C}$  given in Definition 7 states that there is an inductive definition over  $\mathcal{C}$  that is unbounded. As we saw in the preceding section, it is possible to think of inductive definitions as computations over bounded variable element types. Intuitively speaking, for  $\mathcal{C}$  to admit unbounded inductions, it must contain structures with arbitrarily large numbers of types. This motivates a notion of compactness of a class of finite structures, which we define below. The definition of a class  $\mathcal{C}$  being  $k$ -compact is essentially equivalent to McColm's condition for  $\mathcal{C}$  being  $k$ -anti-proficient (see [McC90]).

**Definition 19.** The class of structures  $\mathcal{C}$  is  $k$ -compact, if and only if,  $S_k^k(\mathcal{C})$  is finite.

In other words, a class  $\mathcal{C}$  is  $k$ -compact, if and only if, there are only finitely many  $L^k$ -types of  $k$ -tuples realized in structures in  $\mathcal{C}$ . Observe that if  $S_k^k(\mathcal{C})$  is finite, then  $S_l^k(\mathcal{C})$  is finite for all  $l \leq k$ . The property we have defined is called  $k$ -compactness because it is equivalent to  $\mathcal{C}$  satisfying a certain "compactness condition", as we show next. Recall from Definition 8 that a set of  $L^k$  formulas is an  $L^k$ -type, if and only if, it is the  $L^k$ -type of some tuple in some (finite or infinite) structure.

**Theorem 20.** *A class of finite structures  $\mathcal{C}$  is  $k$ -compact, if and only if, for every  $L^k$ -type  $\tau$ , if for every finite subset  $\delta$  of  $\tau$ , there is a type  $\tau' \in S_l^k(\mathcal{C})$  such that  $\delta \subseteq \tau'$ , then  $\tau \in S_l^k(\mathcal{C})$ .*

**Proof:**

$\Rightarrow$  Let  $\mathcal{C}$  be  $k$ -compact and let  $S_l^k(\mathcal{C}) = \{\tau_1, \dots, \tau_n\}$ . We know from Theorem 9 that there are formulas  $\varphi_1, \dots, \varphi_n$  that isolate the types  $\tau_1, \dots, \tau_n$  respectively. Thus, if  $\tau$  is a type that is not realized in any structure in  $\mathcal{C}$ , it must be the case that  $\neg\varphi_1, \dots, \neg\varphi_n \in \tau$ . But then,  $\{\neg\varphi_1, \dots, \neg\varphi_n\}$  is a finite subset of  $\tau$  that is not realized in any structure in  $\mathcal{C}$ .

$\Leftarrow$  Suppose  $\mathcal{C}$  is not  $k$ -compact. Let  $S_k^k(\mathcal{C}) = \{\tau_i \mid i \in \omega\}$  and let  $\varphi_i (i \in \omega)$  be an enumeration of formulas such that  $\varphi_i$  isolates  $\tau_i$ . Let  $\Gamma = \{\neg\varphi_i \mid i \in \omega\}$ . We show that  $\Gamma$  can be completed to a type  $\tau$  such that every finite subset of  $\tau$  is realized in some structure in  $\mathcal{C}$ . However, it is clear that  $\tau$  could not be realized in any structure in  $\mathcal{C}$ .

To construct  $\tau$ , let  $\alpha_i (i \in \omega)$  be a fixed enumeration of all formulas of  $L^k$ . We define the sets of formulas  $\delta_n$  inductively as follows:

$$\begin{aligned} \delta_0 &= \emptyset \\ \delta_{n+1} &= \begin{cases} \delta_n \cup \{\alpha_n\} & \text{if } \delta_n \cup \{\alpha_n\} \subseteq \tau_i \text{ for infinitely many } i \in \omega; \\ \delta_n \cup \{\neg\alpha_n\} & \text{otherwise.} \end{cases} \end{aligned}$$

A simple argument by induction shows that for all  $n$ ,  $\delta_n \subseteq \tau_i$  for infinitely many  $i$ . Let  $\tau = \Gamma \cup \bigcup_{n \in \omega} \delta_n$ . The construction ensures that every finite subset of  $\tau$  is realized in some structure in  $\mathcal{C}$ . It then follows from a direct application of the Compactness Theorem that  $\tau$  is realized in some (possibly infinite) structure. Thus,  $\tau$  is an  $L^k$ -type, and as was observed earlier, it cannot be realized in any structure in  $\mathcal{C}$ . ■

We motivated the definition of  $k$ -compactness with the intuition that inductions are bounded over a class of structures if there is a bound on the number of types that are realized in any structure in the class. However,  $k$ -compactness is, on the face of it, a stronger condition. It stipulates that there is a finite number of types realized in the entire class. The next lemma shows that the two notions, indeed, coincide.

**Lemma 21.** *For any class of finite structures  $\mathcal{C}$ , the following conditions are equivalent:*

1.  $\mathcal{C}$  is  $k$ -compact;
2.  $\sup(\{\nu_k(\mathfrak{A}) \mid \mathfrak{A} \in \mathcal{C}\}) < \omega$ ; and
3.  $\sup(\{\text{sr}^k(\mathfrak{A}) \mid \mathfrak{A} \in \mathcal{C}\}) < \omega$ .

**Proof:**

1  $\Rightarrow$  2 It is clear that  $\nu_k(\mathfrak{A}) \leq \text{card}(S_k^k(\mathcal{C}))$  for all  $\mathfrak{A} \in \mathcal{C}$ . Thus, if  $S_k^k(\mathcal{C})$  is finite, there is a finite bound on all  $\nu_k(\mathfrak{A})$ .

2  $\Rightarrow$  3 This follows immediately from Lemma 12.

3  $\Rightarrow$  1 It follows from the definition of Scott rank that every  $L^k$ -type realized in  $\mathfrak{A}$  is isolated by a formula of  $L^k$  of quantifier rank at most  $\text{sr}^k(\mathfrak{A})$ . Thus if  $m = \sup(\{\text{sr}^k(\mathfrak{A}) \mid \mathfrak{A} \in \mathcal{C}\})$ , every type in  $S_k^k(\mathcal{C})$  is isolated by a formula of quantifier rank at most  $m$ . However, for any fixed  $m$ , there are, up to logical

equivalence, only finitely many formulas of  $L^k$  of quantifier rank at most  $m$ .<sup>4</sup> Thus,  $S_k^k(\mathcal{C})$  must be finite. ■

We can now relate closure ordinals of formulas and types through the following lemma, which will then allow us to make the connection between proficiency and  $k$ -compactness in Theorem 23 below.

**Lemma 22.** *For every  $R$ -positive formula  $\varphi \in L^k$  and every finite structure  $\mathfrak{A}$ ,  $\|\varphi\|_{\mathfrak{A}} \leq \nu_{2k}(\mathfrak{A})$ .*

**Proof:**

Each stage  $\varphi^m$  of the iteration of the operator defined by  $\varphi$  is closed under the equivalence relation  $\equiv^{2k}$  (see Theorem 6); therefore, it can be viewed as a union of equivalence classes under this relation. Furthermore, since the operator defined by  $\varphi$  is monotone, the number of stages in which it converges must be bounded by the number of equivalence classes. This number is, of course, just  $\nu_{2k}(\mathfrak{A})$ . ■

We are now in a position to prove the following theorem.

**Theorem 23.** *Let  $\mathcal{C}$  be a class of finite structures of signature  $\sigma$ .  $\mathcal{C}$  is proficient, if and only if, there is a  $k$  such that  $\mathcal{C}$  is not  $k$ -compact.*

**Proof:**

Suppose  $\mathcal{C}$  is proficient. Then, there is a  $k$  and a formula  $\varphi \in L^k$  such that  $\sup(\{\|\varphi\|_{\mathfrak{A}} \mid \mathfrak{A} \in \mathcal{C}\}) \geq \omega$ . But it then follows immediately by Lemmas 22 and 21 that  $\mathcal{C}$  is not  $2k$ -compact.

For the other direction, let  $k$  be such that  $\mathcal{C}$  is not  $k$ -compact. By Lemma 21, it follows that  $\sup(\{\text{sr}^k(\mathfrak{A}) \mid \mathfrak{A} \in \mathcal{C}\}) \geq \omega$ . From this and Lemma 17 it follows at once that  $\mathcal{C}$  is proficient. In particular,  $\sup(\{\|\zeta\|_{\mathfrak{A}} \mid \mathfrak{A} \in \mathcal{C}\}) \geq \omega$ , where  $\zeta$  is the formula defined above with respect to  $k$  and  $\sigma$ . ■

Having related the notions of  $k$ -compactness and proficiency in Theorem 23, we now establish the relationship between  $k$ -compactness of a class  $\mathcal{C}$  and the expressive power of  $L_{\infty\omega}^k$  over this class, in the following theorem.

**Theorem 24.** *Let  $\mathcal{C}$  be a class of finite structures.*

1. *If  $\mathcal{C}$  is  $k$ -compact, then only finitely many distinct queries are definable in  $L_{\infty\omega}^k$  over  $\mathcal{C}$ . Moreover, each such query is already definable in  $L^k$ .*
2. *If  $\mathcal{C}$  is not  $k$ -compact, then  $2^\omega$  distinct queries are definable in  $L_{\infty\omega}^k$  over  $\mathcal{C}$ . Hence, some such query is not first-order definable.*

**Proof:**

1. Suppose  $\mathcal{C}$  is  $k$ -compact. We know from Theorem 9 that there is a list  $\varphi_1, \dots, \varphi_n$  of  $L^k$  formulas which isolates each of the  $L^k$ -types of  $k$ -tuples realized over structures in  $\mathcal{C}$ . Clearly, every  $L_{\infty\omega}^k$  query is equivalent over  $\mathcal{C}$  to a disjunction of the  $\varphi_i$ 's. But there are  $2^n$  such disjunctions and each of them is a formula of  $L^k$ .

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<sup>4</sup> Recall that we are dealing with purely relational languages. This is not true in languages that include function symbols.

2. Suppose  $\mathcal{C}$  is not  $k$ -compact. Again we know from Theorem 9 that there is a list  $\varphi_i (i \in \omega)$  of formulas of  $L^k$  which isolate the countably many distinct types realized over structures in  $\mathcal{C}$ . Again, each  $L_{\infty\omega}^k$  query is equivalent over  $\mathcal{C}$  to a (countable) disjunction of the  $\varphi_i$ 's. But there are  $2^\omega$  such disjunctions (which define distinct queries) and only countably many first-order formulas.

■

We now have the positive solution to McColm's second conjecture as a corollary of Theorems 23 and 24.

**Corollary 25** [McC90],[KV92a]. *A class  $\mathcal{C}$  of finite structures is proficient, if and only if, there is a query expressible over  $\mathcal{C}$  in  $L_{\infty\omega}^\omega$  that is not expressible in FO on  $\mathcal{C}$ .*

Indeed, we have also shown a somewhat stronger result. It is a direct consequence of Theorem 24 that for every  $k$ ,  $L_{\infty\omega}^k$  collapses to FO on a class of finite structures  $\mathcal{C}$ , if and only if,  $L_{\infty\omega}^k$  collapses to  $L^k$  on  $\mathcal{C}$ . This is a version of what Kolaitis and Vardi termed the "ramified" version of McColm's conjecture [KV92a].

The proof of Theorem 23 relies on the fact that in any class that is not  $k$ -compact, the induction defined by the formula  $\zeta$  is unbounded. As we see below, we can extract from this fact an LFP definable query that is closed under the relation  $\equiv^k$  but is not definable in  $L^k$  in any class that is not  $k$ -compact. The query is constructed to include exactly one  $\equiv^k$ -equivalence class in each structure  $\mathfrak{A}$ . The equivalence class selected will be one of maximal Scott rank in  $\mathfrak{A}$ . This is formally stated in the lemma below.

**Lemma 26.** *For any  $k$  there is a formula  $\chi(x_1, \dots, x_k)$  of LFP with the following properties: for every structure  $\mathfrak{A}$ ,*

1.  $\mathfrak{A} \models \exists x_1, \dots, x_k \chi$ ;
2. for any two  $k$ -tuples  $s$  and  $s'$  of elements of  $\mathfrak{A}$ , if  $\mathfrak{A} \models \chi[s]$  and  $\mathfrak{A} \models \chi[s']$ , then  $\text{Type}_k(\mathfrak{A}, s) = \text{Type}_k(\mathfrak{A}, s')$ ;
3.  $\chi$  is equivalent to a formula of  $L_{\infty\omega}^k$ ;
4. for every  $k$ -tuple  $s$  of elements of  $\mathfrak{A}$ , if  $\mathfrak{A} \models \chi[s]$ , then  $\text{sr}_{\mathfrak{A}}^k(s) = \text{sr}^k(\mathfrak{A})$ .

**Proof:**

Let  $\zeta(R, z_1, \dots, z_{2k})$  be the formula given by Theorem 15. Consider the stage comparison relation of this formula,  $\leq_\zeta$ , which is definable in LFP by Theorem 2. Define  $\theta(x_1, \dots, x_k)$  as follows:

$$\theta \equiv \exists y_1, \dots, y_k \forall z_1, \dots, z_{2k} (\text{lfp}(R, z_1, \dots, z_{2k}) \zeta(z_1, \dots, z_{2k}) \rightarrow \langle z_1, \dots, z_{2k} \rangle \leq_\zeta \langle x_1, \dots, x_k, y_1, \dots, y_k \rangle).$$

Then, by Lemma 17, for any structure  $\mathfrak{A}$  and any  $k$ -tuple  $s$  of elements of  $\mathfrak{A}$ ,  $\mathfrak{A} \models \theta[s]$ , if and only if,  $\text{sr}_{\mathfrak{A}}^k(s) = \text{sr}^k(\mathfrak{A})$ . That is  $\theta$  picks out all the tuples in  $\mathfrak{A}$  of maximal Scott rank. Since there must clearly be some such tuples,  $\theta$  satisfies the first and the fourth conditions. Furthermore, since tuples that realize the

same type have the same Scott rank, the query defined by  $\theta$  is closed under the equivalence relation  $\equiv^k$ , and therefore it is definable in  $L_{\infty\omega}^k$ , and it satisfies the third condition. In general, however, it does not satisfy the second condition, since there may be more than one equivalence class of maximal Scott rank in any given structure. To select from among these, we use the ordering on equivalence classes,  $<_k$  given by Theorem 18. Now, define the formula  $\chi(x_1, \dots, x_k)$  as follows:

$$\chi \equiv \theta(x_1, \dots, x_k) \wedge \forall y_1, \dots, y_k (\theta(y_1, \dots, y_k) \rightarrow \neg(y_1, \dots, y_k) <_k (x_1, \dots, x_k)).$$

Since  $\chi$  selects exactly one equivalence class, it satisfies condition 1 and 2, and the equivalence class is selected from among those selected by  $\theta$ , so it satisfies condition 4. Since the entire equivalence class is chosen (this follows from the definition of the pre-order  $<_k$ ),  $\chi$  defines a query closed under the equivalence relation  $\equiv^k$  and it therefore satisfies condition 3. ■

It is clear that the formula  $\chi$  is not equivalent to any formula of  $L^k$  in any class  $\mathcal{C}$  that is not  $k$ -compact. Indeed, suppose it were equivalent to such a formula of quantifier rank  $m$ . Then, since  $\mathcal{C}$  is not  $k$ -compact, it contains a structure  $\mathfrak{A}$  with  $\text{sr}^k(\mathfrak{A}) > m$ , but all tuples  $s$  in  $\mathfrak{A}$  such that  $\mathfrak{A} \models \chi[s]$  are  $L^k$ -equivalent, and by the definition of Scott ranks, they cannot be distinguished from all other tuples in  $\mathfrak{A}$  by formulas of quantifier rank  $\leq m$ , yielding a contradiction. This argument enables us to establish the following two theorems:

**Theorem 27.** *For any class of structures  $\mathcal{C}$ , the following are equivalent:*

1.  $\mathcal{C}$  is  $k$ -compact.
2.  $L_{\infty\omega}^k \cap \text{LFP} = L^k$  on  $\mathcal{C}$ .

**Proof:**

(1) $\Rightarrow$ (2) follows from Theorem 24. Conversely, if (1) is false, then the formula  $\chi$  of Theorem 26 witnesses that the separation of  $L_{\infty\omega}^k \cap \text{LFP}$  from  $L^k$ . ■

The above can be seen as strengthening Theorem 24 in the sense that it shows that if  $\mathcal{C}$  is not  $k$ -compact, then not only can we separate  $L_{\infty\omega}^k$  from  $L^k$ , but the separating query can be chosen to be LFP definable.

**Definition 28.** A class of structures  $\mathcal{C}$  has the  *$k$ -preservation property* if every query that is  $\equiv^k$ -closed over  $\mathcal{C}$  and first order definable on  $\mathcal{C}$  is definable in  $L^k$  over  $\mathcal{C}$ .

This definition allows us to state a sufficient condition on a class of structures for the separation of LFP and FO.

**Theorem 29.** *If there is a  $k$  such that  $\mathcal{C}$  is not  $k$ -compact and has the  $k$ -preservation property, then LFP does not collapse to FO on  $\mathcal{C}$ .*

## 4 The Ordered Conjecture

Theorem 29 raises the question of which classes of structures  $\mathcal{C}$  have the  $k$ -preservation property. In this section, we investigate this question for classes of ordered structures. We also show that this is linked to the question of whether the class of all finite structures  $\mathcal{F}_\sigma$  has the  $k$ -preservation property.

In the case of the class of all structures (finite or infinite), this question is resolved as a direct consequence of a result proved by Immerman and Kozen [IK89], using the compactness theorem. This is stated in the theorem below.

**Theorem 30 [IK89].** *The class  $\mathcal{S}$  of all structures (finite or infinite) has the  $k$ -preservation property, for all  $k$ .*

It has been observed that most preservation theorems that hold on the class of all structures fail when we restrict ourselves to finite structures (see [Gur84]). One would expect that this is the case for the above as well. Here, we show that the question of whether such a preservation theorem holds on finite structures is connected to Conjecture 1. To see this, we first establish a technical lemma.

For any signature  $\sigma$ , let the width of  $\sigma$ , denoted  $w(\sigma)$ , be the maximum arity of any relation symbol in  $\sigma$ . Fix a signature  $\sigma$  and let  $m = \max(w(\sigma), 3)$ . We then have the following:

**Lemma 31.** *For any structure  $\mathfrak{A}$  in  $\mathcal{O}_\sigma$ , and any  $l$ -tuple  $s = \langle a_1, \dots, a_l \rangle$  of elements in  $\mathfrak{A}$ , where  $l \leq m$ , there is a formula  $\varphi$  of  $L^m$  such that, for any structure  $\mathfrak{B}$  of signature  $\sigma \cup \{\leq\}$ ,  $\mathfrak{B} \models \varphi[t]$  if and only if there is an isomorphism  $f : \mathfrak{A} \cong \mathfrak{B}$  with  $f(s) = t$ .*

**Proof:**

We first show that, for every element  $a$  of  $\mathfrak{A}$ , there is a formula  $\beta_a(x)$  of  $L^2$  such that  $a$  is the unique element of  $\mathfrak{A}$  satisfying  $\mathfrak{A} \models \beta_a[a]$ . To show this, we inductively define the following class of formulas.

$$\begin{aligned} \alpha_0(x) &\equiv \neg(x = x) \\ \alpha_{n+1}(x) &\equiv \forall y((y \leq x) \rightarrow (x = y \vee \exists x(x = y \wedge \alpha_n(x)))) \end{aligned}$$

It is clear that  $\mathfrak{A} \models \alpha_n[a]$  if and only if there are at most  $n$  elements less than or equal to  $a$  in the linear order  $\leq^{\mathfrak{A}}$ . Thus, the formula  $\beta_n \equiv \neg\alpha_{n-1} \wedge \alpha_n$  identifies the  $n$ th element of the order uniquely.

Using these formulas, it is clear that any  $m$ -tuple can be uniquely identified by a formula of  $L^m$ , and we can therefore construct a sentence  $\psi_{\mathfrak{A}}$  of  $L^m$  that determines the structure  $\mathfrak{A}$  up to isomorphism among structures in  $\mathcal{O}_\sigma$ . If  $\lambda$  is the sentence of  $L^3$  that asserts that  $\leq$  is a linear order, then  $\psi_{\mathfrak{A}} \wedge \lambda \wedge \bigwedge_{1 \leq i \leq l} \beta_{a_i}(x_i)$  is the required formula  $\varphi$ . ■

It follows from Lemma 31 that if  $\mathcal{C}$  is a class of ordered structures over some signature  $\sigma$ , where  $w(\sigma) \leq m$ , then every query, of arity at most  $m$ , on  $\mathcal{C}$  is definable in  $L_{\infty\omega}^m$  (assuming  $m$  is at least 3). Furthermore, if  $\varphi$  is any first-order formula (with at most  $k$  free variables, for any  $k \geq m$ ) in such a signature  $\sigma$  and  $\lambda$  is as above, then it follows easily from Lemma 31 that  $\varphi \wedge \lambda$  is equivalent over

the class  $\mathcal{F}_\sigma$  to a formula of  $L_{\infty\omega}^k$ . Let  $\sigma'$  denote the signature  $\sigma \cup \{\leq\}$ . We can now prove the following theorem.

**Theorem 32.** *If there is a  $k \geq m$  such that  $\mathcal{F}_{\sigma'}$  has the  $k$ -preservation property, then every class  $\mathcal{C} \subseteq \mathcal{O}_\sigma$  has the  $k$ -preservation property.*

**Proof:**

Let  $\varphi$  be any first-order formula with free variables among  $x_1, \dots, x_k$ . Since  $m \leq k$ , by the observations above,  $\varphi \wedge \lambda$  is equivalent over  $\mathcal{F}_{\sigma'}$  to a formula of  $L_{\infty\omega}^k$ . But then, by the  $k$ -preservation property of  $\mathcal{F}_{\sigma'}$ , there is a formula  $\psi$  of  $L^k$  that is equivalent to  $\varphi \wedge \lambda$  over  $\mathcal{F}_{\sigma'}$ . Since  $\lambda$  is true in all structures in  $\mathcal{C}$ , it follows that on  $\mathcal{C}$ ,  $\psi$  defines the same query as  $\varphi$ . ■

Theorem 32 shows that a preservation theorem along the lines of Theorem 30 for finite structures would resolve Conjecture 1. This, however, seems an unlikely eventuality, since it seems unlikely that every class of ordered structures has the  $k$ -preservation property for some  $k$ . This is because, for any class  $\mathcal{C} \subseteq \mathcal{O}_\sigma$  and any  $k \geq m$ , if  $\mathcal{C}$  has the  $k$ -preservation property, then every first order definable query of arity  $k$  or less is definable in  $L^k$ . Thus, in particular, every first order sentence is equivalent to one with no more than  $k$  variables. Nonetheless, there are interesting classes of structures for which this property holds. The following result is due to Poizat [Poi82] (for another exposition of this result see [IK89]).

**Theorem 33 [Poi82].** *If  $\sigma$  contains only unary relation symbols, then every first order formula with at most three free variables is equivalent on  $\mathcal{O}_\sigma$  to a formula of  $L^3$ .*

As a corollary, we get the following theorem.

**Theorem 34.** *For any unary signature  $\sigma$ , and any class  $\mathcal{C} \subseteq \mathcal{O}_\sigma$ , if  $\mathcal{C}$  contains arbitrarily large structures, then LFP does not collapse to FO on  $\mathcal{C}$ .*

## 5 Complexity Theoretic Implications

It turns out that a resolution of Conjecture 1, whether positive or negative, would have important implications in complexity theory. Moreover, if the question is resolved by the methods outlined in the previous section, i.e. by showing that the class  $\mathcal{O}_\sigma$  has the  $k$ -preservation property for some  $k$ , then this has some unlikely implications, that follow from the observation contained in the next proposition.

**Proposition 35.** *If  $\mathcal{O}_\sigma$  has the  $k$ -preservation property, then every first order definable  $k$ -ary query on  $\mathcal{F}_\sigma$  is computable in  $\text{DTIME}[n^k]$ .*

**Proof:**

By the  $k$ -preservation property, every first order definable  $k$ -ary query is definable by a formula  $\varphi$  of  $L^k$ . In such a formula, every sub-formula contains at most  $k$  free variables. Since there is a constant number of such sub-formulas, we can evaluate  $\varphi$  in a structure  $\mathfrak{A}$  of size  $n$ , by enumerating all  $n^k$   $k$ -tuples in  $\mathfrak{A}$ , and

checking whether they satisfy the sub-formulas. It can be verified that such an algorithm runs in time  $O(n^k)$ . ■

Taking  $\sigma$  to be the language of graphs, i.e. the signature consisting of just one binary relation, it follows from the above that if there is a  $k$  such that  $\mathcal{O}_\sigma$  has the  $k$ -preservation property, then for every  $c$ , the problem of determining whether a graph has a  $c$ -clique is solvable in  $\text{DTIME}[n^k]$ . On the other hand, it is difficult to prove that there is no  $k$  such that every first order definable Boolean query on  $\mathcal{F}_\sigma$  is computable in  $\text{DTIME}[n^k]$ , because such a result would imply the separation of  $\text{PTIME}$  from  $\text{PSPACE}$  (see [ST95]).

Moreover, if we could show that Conjecture 1 is false, that would also establish the separation of  $\text{PTIME}$  and  $\text{PSPACE}$ . This follows from the result in [DH95] that on any infinite class of ordered structures, there is a PFP query that is not first order definable. Thus, we have the following proposition.

**Proposition 36.** *If there is an infinite class of ordered structures on which  $\text{LFP} = \text{FO}$ , then  $\text{PTIME} \neq \text{PSPACE}$ .*

In order to state the complexity theoretic implications of a positive resolution of Conjecture 1, we introduce some notation.  $\text{Log-H}$  denotes the logarithmic time hierarchy, i.e. the class of those problems that can be solved in logarithmic time by an alternating machine with a bounded number of alternations. Similarly,  $\text{Lin-H}$  denotes the linear time hierarchy, i.e. those problems that can be solved by a linear time, bounded depth, alternating machine.

Consider a signature  $\sigma$  including ternary relation symbols  $+$  and  $\times$ . Let  $\mathcal{C} \subseteq \mathcal{O}_\sigma$  be the class of structures such that  $+$  is interpreted as the addition relation consistent with the order  $\leq$ , and  $\times$  is interpreted as the corresponding multiplication relation. It follows from a result of Barrington *et al.* [BIS90] that  $\text{FO} = \text{Log-H}$  on this class of structures. Now consider the class of structures  $\mathcal{D}$  of the form  $\langle m, +, \times \rangle$ , i.e. containing no relations other than the numerical predicates. This allows us to give a succinct representation of these structures. That is, since the structure is completely determined by the value of  $m$ , we can represent it as a binary string of length  $\log(m)$ . It then follows that on this class, a query is definable in first order logic if, and only if, it is in  $\text{Lin-H}$  (another way to characterize this class is as the class  $\text{RUD}$  of rudimentary sets of binary strings, which was shown in [Wra79] to be equivalent to  $\text{Lin-H}$ ). Similarly, a query is definable in  $\text{LFP}$  on this class if and only if it is computable in  $\text{DTIME}[2^{\mathcal{O}(n)}]$  (note here that  $n = \log(m)$  is the length of the binary string). We write  $\text{ETIME}$  to denote the latter class. Thus, we have the following proposition.

**Proposition 37.** *If Conjecture 1 holds then  $\text{Lin-H} \neq \text{ETIME}$ .*

The complexity theoretic separation of Proposition 37 can be seen as a linear counterpart to the separation of  $\text{PH}$  from  $\text{EXPTIME}$ .

## 6 Conclusions

To conclude, we present several directions of investigation suggested by the results we have presented. The first is to show that the class of ordered graphs

does not have the  $k$ -preservation property for any  $k$ , or equivalently, to show that there is a class of ordered structures for which FO does not collapse to  $L^k$ , for any  $k$ . Another direction is to investigate for what classes of ordered structures the sufficient condition provided by Theorem 29 can be used to establish the separation of LFP and FO. That is, for what classes of ordered structures is it the case that there is a  $k$  such that FO collapses to  $L^k$ ? We showed that this is true for all classes of strings (i.e. linear orders with additional unary predicates), but are there other interesting classes of structures for which this holds? Since we do not expect all classes of ordered structures to have this property, it would also be instructive to find other, weaker, sufficient conditions on a class of ordered structures so that  $LFP \neq FO$ .

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