# ELEMENTARY PROPERTIES OF THE FINITE RANKS

ANUJ DAWAR, KEES DOETS, STEVEN LINDELL, AND SCOTT WEINSTEIN

Abstract. This note investigates the class of finite initial segments of the cumulative hierarchy of pure sets. We show that this class is first-order definable over the class of finite directed graphs and that this class admits a first-order definable global linear order. We apply this last result to show that FO(<, BIT) = FO(BIT).

This note establishes some elementary properties of the finite initial segments of the cumulative hierarchy of pure sets. We define the sets $V_n, n \in \omega$ by induction as follows: $V_0 = \emptyset$; $V_{n+1} = \mathcal{P}(V_n)$. (Here, $\mathcal{P}(X)$ is the *power set of* $X$, that is, the set of all subsets of $X$.) The collection of nonempty finite ranks, $\mathcal{FR}$, is $\{V_n : n > 0\}$. We will often use $V_n$ to denote the directed graph whose set of nodes is $V_n$ and whose edge relation is $\in$, the set membership relation; similarly, we will use $\mathcal{FR}$ to denote the collection of such graphs $V_n$ for $n > 0$. More generally, if $M$ is a collection of sets, we will also use $M$ to denote the directed graph whose set of nodes is $M$ and whose edge relation is set membership restricted to $M$. We will use $\in$ both as the symbol for the edge relation of the first-order language of directed graphs and as the symbol for the set membership relation.

## 1. First-Order Definability

The purpose of this section is to establish the following elementary property of $\mathcal{FR}$.

**Theorem 1.1.** *The class $\mathcal{FR}$ is first-order definable over the class of finite directed graphs, that is, there is a first-order sentence $\Phi$ such that for every finite directed graph $A$,*

$$A \models \Phi, \text{ iff, } (\exists n > 0)(A \cong V_n).$$

We will construct the sentence $\Phi$ required by Theorem 1.1 as a conjunction of five first-order sentences. We list the first four of these immediately and begin to establish a few lemmas about directed graphs which satisfy their conjunction. We abbreviate the formula $\exists u(x \in u)$ by $\mathrm{Bd}(x)$ and we say that an object which satisfies this formula is bounded.

1. $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$ (Extensionality)
2. $\forall x(\exists y(y \in x) \rightarrow \exists y(y \in x \wedge \forall z(z \in x \rightarrow z \notin y)))$ (Foundation)
3. $\forall x \forall y(\mathrm{Bd}(y) \rightarrow \exists z \forall w(w \in z \leftrightarrow (w \in x \vee w = y)))$ (Bounded Adjunction)
4. $\exists x \forall y(y \notin x)$ (Empty Set)

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We use $\Psi$ to abbreviate the conjunction of conditions 1-4. It is easy to verify that for every $n > 0, V_n \models \Psi$.

**Lemma 1.2.** *If $A \models$ the conjunction of conditions 3 and 4 above, then for each $n \in \omega$,*

$$A \models \forall x_1 \ldots \forall x_n ( \bigwedge_{1 \leq i \leq n} \mathrm{Bd}(x_i) \rightarrow \exists y \forall z (z \in y \leftrightarrow \bigvee_{1 \leq i \leq n} z = x_i)).$$

*Proof.* The proof is by induction on $n$. For $n = 1$, conditions 3 and 4 together easily imply the existence of all singletons whose members are bounded. The induction step is similarly a direct consequence of condition 3. ■

**Lemma 1.3.** *If $A \models$ the conjunction of conditions 2, 3, and 4, then $A$ is acyclic.*

*Proof.* Suppose, for *reductio ad absurdum*, that $A$ contains a cycle $a_1 \in a_n \in \ldots \in a_1$. Since each $a_i$ is bounded, by Lemma 1.2 there is a point $a$ in $A$ whose members are exactly $a_1, \ldots, a_n$. But $a$ does not have an $\in$-minimal element. This contradicts the hypothesis that $A$ satisfies condition 2. ■

Recall that a set $M$ is transitive, if and only if, $\forall x(x \in M \rightarrow \forall y(y \in x \rightarrow y \in M)$.

**Corollary 1.4.** *If $A$ is finite and $A \models \Psi$, then there is a transitive set $M$ such that $A \cong M$.*

*Proof.* Since $A$ is finite, by Lemma 1.3, $A$ is well-founded. The corollary now follows immediately from the Mostowski Collapsing Lemma (see [Kun80]) which states that any well-founded model of the axiom of extensionality is isomorphic to a transitive set. ■

REMARK: The finite models of $\Psi$ (conditions 1–4) admit the following simple characterization: *a finite non-empty transitive set $M$ is a model of $\Psi$ iff there exists a (necessarily: transitive) set $N$ such that $M = \mathcal{P}(N)$.*

*Proof.* It is easy to verify that if $N$ is a finite transitive set, then $\mathcal{P}(N) \models \Psi$. For the converse, let $N = \bigcup M$ (= the union of all elements of $M$). Then, (i) $\mathcal{P}(N) \subseteq M$ is immediate from Lemma 1.2 and (ii) $M \subseteq \mathcal{P}(N)$ is immediate from the definition of $N$. ■

It only remains to add a fifth condition to guarantee that our finite transitive set is indeed a full rank. The above characterization of finite models of $\Psi$ suggests we could require that not only $M$ is the powerset of a set $N$, but also, that $N$ be a powerset of some $N'$, that $N'$ be a powerset of some $N''$, etc., until, eventually, one arrives at $\emptyset$. Our fifth condition is a way to make this precise.

**Lemma 1.5.** *Suppose that $r$ is a set of finite sets such that*

[*]: $\quad \forall x \in r(x \neq \emptyset \rightarrow \exists y \in r(x = \mathcal{P}(y)))$.

*Then every element of $r$ is a (finite) rank.*

*Proof.* Straightforward induction on the number of elements in an element of $r$. ■

Now, let condition 5 be (copying [*] from Lemma 1.5):

5. $\exists r( \text{[*]} \wedge \exists u(u = \bigcup_{z \in r} \mathcal{P}(z) \wedge \forall x(x \subseteq u)))$.

*Proof of Theorem 1.1.* First, to see that $V_{n+1} \models 5$, note that the choice of $r = \{V_0, \ldots, V_{n-1}\}$ and $u = V_n$ satisfies 5 in $V_{n+1}$.

Next, let $M$ be a finite transitive model of 1–5. Suppose that $r \in M$ and $u \in M$ witness satisfaction of 5 in $M$. Since by Lemma 1.2, $M$ contains every subset of its elements, powersets are *absolute* (a powerset relative to $M$ is a real powerset), and Lemma 1.5 applies. Thus, elements of $r$ are ranks, and for some number $n$, $u = \bigcup_{z \in r} \mathcal{P}(z) = V_n$, and hence $M \subseteq V_{n+1}$. Also, $V_{n+1} \subseteq M$; for, if $x \in V_{n+1}$, i.e., $x \subseteq V_n$, then, by Lemma 1.2, $x \in M$ follows. ∎

REMARKS: (i) Condition [*] in Lemma 1.5 can be viewed as a "post-fixed point" condition on the set $r$; the monotone operator involved here maps a set $r$ to $\{\emptyset\} \cup \{\mathcal{P}(y) \mid y \in r\}$.

(ii) For a possibly simpler axiomatization, delete conditions 2 (Foundation), 3 (Bounded Adjunction) and 4 (Empty Set); but then in condition 5 two changes have to be performed:

(a) consider the two occurrences of the powerset operation as *defined* by

$$Y = \mathcal{P}(x) \text{ iff } \forall u \in Y (u \subseteq x) \wedge \emptyset \in Y \wedge \forall u \in Y \forall v \in x (u \cup \{v\} \in Y)$$

and

(b) strengthen $\forall x(x \subseteq u)$ to "$\mathcal{P}(u)$ equals the universe". (Note that, in the above definition, $Y$ is allowed to be a class.)

This definition of powersets is both correct in the finite realm and absolute, allowing the deletion of Bounded Adjunction (which was responsible for absoluteness). The strengthening of $\forall x(x \subseteq u)$ is needed to eliminate the second appeal to Lemma 1.2 in the proof of Theorem 1.1.

This reduces the axiomatization to just *two* axioms. (A proof resembling that of Theorem 1.1, that every finite model of these two axioms is a rank, is probably best carried out in a non-well-founded "meta-environment" in which *every* model of Extensionality —be it well-founded or not— has a transitive isomorph.)

(iii) Let $L$ be the first-order language of set theory. Recall that a sentence of $L$ is *finitely valid*, if and only if, it is true in every $L$-structure whose universe is finite. Let $\mathrm{Val}_f$ be the collection of finitely valid sentences of $L$. The sentence $\Psi$ (introduced immediately before Lemma 1.2) can be used to give a short proof of

TRAKHTENBROT'S THEOREM [Tra50]: $\mathrm{Val}_f$ is co-r.e. complete.

*Proof.* It is clear that $\mathrm{Val}_f$ is co-r.e. Thus, in order to establish the result, it suffices to show that $\mathrm{Val}_f$ is co-r.e. hard.

Let $\varphi(x)$ be a $\Pi_1$ formula with one free variable (see [Bar75] for the notion of a $\Pi_1$ formula of set theory). It is easy to verify the following important fact:

(1.1)

$$\forall a \in V_\omega (V_\omega \models \varphi[a] \iff \forall A \subseteq V_\omega (a \in A \text{ and } A \text{ is finite and } A \text{ is transitive } \to \mathcal{P}(A) \models \varphi[a])).$$

Now let $X \subseteq \omega$ be a co-r.e. set. We need to show that there is a computable function $k \mapsto \varphi_k$ from $\omega$ to $L$ such that for all $k \in \omega$,

$$k \in X \iff \varphi_k \in \mathrm{Val}_f.$$

We use the fundamental theorem that $Y \subseteq \omega$ is co-r.e., if and only if, $Y$ is $\Pi_1$-definable over $V_\omega$ (see [Bar75], Theorem 2.3). So let $\chi(x)$ be a $\Pi_1$ formula which defines $X$ over $V_\omega$, that is,

$$\forall a \in V_\omega (a \in X \iff V_\omega \models \chi[a]).$$

For each $k \in \omega$, let $\overline{k}(x)$ be a formula of set theory such that for any transitive set $A$,

$$\forall a \in A((A \models \overline{k}[a]) \iff k = a).$$

Finally, let

$$\varphi_k = \Psi \to \forall x(\overline{k}(x) \to \chi(x)).$$

Now 1.1 and the Remark following Corollary 1.4 may be used to verify the following sequence of equivalences which conclude the proof. For all $k \in \omega$,

$$\begin{aligned} k \in X \quad &\iff \quad V_\omega \models \chi[k] \\ &\iff \quad \text{for every finite transitive set } A, A \models \forall x(\overline{k}(x) \to \chi(x)) \\ &\iff \quad \varphi_k \in \mathrm{Val}_f. \end{aligned}$$

■

## 2. Global Ordering

In this section, we establish that there is a first-order definable global linear order on a collection of finite structures which includes $\mathcal{FR}$. Let $V_\omega$ be the collection of hereditarily finite sets, that is, $V_\omega = \bigcup_{n\in\omega} V_n$. We write $\prec$ to denote the inductively defined linear order on $V_\omega$ which is characterized by the following equivalence:

$$x \prec y \leftrightarrow \exists y' \in (y - x)\forall x' \in (x - y)(x' \prec y').$$

(See below, immediately before Corollary 2.3, for the more usual description of $\prec$.) The following observations are immediate.

1. $\prec$ has order type $\omega$.
2. Each $V_n$ is a $\prec$-initial segment of $V_\omega$.

We say that $X \subseteq V_\omega$ is a *near finite rank*, if and only if, $(\exists n \in \omega)(V_n \subset X \subseteq V_{n+1})$. We use $\mathcal{NFR}$ to denote the collection of near finite ranks. Note that $\mathcal{NFR}$ (properly) contains the collection of all proper $\prec$-initial segments of $V_\omega$.

**Theorem 2.1.** *There is a first-order definable global linear order on $\mathcal{NFR}$. Indeed, there is a first-order formula $\Lambda(x, y)$ such that for every $M \in \mathcal{NFR}$, and for all $a, b \in M$, $M \models \Lambda[a, b]$, if and only if, $a \prec b$.*

*Proof.* Write $\langle x, y\rangle$ for the set-theoretic pairing function $\{\{x\}, \{x, y\}\}$ and $\mathrm{rel}(r)$ for the formula which states that $r$ is a binary relation, that is, a set of ordered pairs. Suppose that $M \in \mathcal{NFR}$; say, $V_n \subset M \subseteq V_{n+1}$.

For a set $r$ of ordered pairs in $M$, define

$$\tau(r) = \{\langle x, y\rangle \in M \mid \exists y' \in (y - x)\forall x' \in (x - y)(\langle x', y'\rangle \in r)\}.$$

Obviously, $\tau$ is a monotone operator over $M$. (In fact, considered as a collection of ordered pairs, $\prec$ is the unique fixed point of $\tau$.)

**Claim 2.2.** If $r \subseteq \tau(r)$, then $r$ is included in $\prec$.

*Proof of Claim 2.2.* Suppose that $\langle x, y\rangle \in r$. We show that $x \prec y$, using $\in$-induction w.r.t. $y$. Since $r \subseteq \tau(r)$, we have that for some $y' \in (y - x)$, $\forall x' \in (x - y)(\langle x', y'\rangle \in r)$. By induction hypothesis, $\forall x' \in (x - y)(x' \prec y')$. Therefore, $x \prec y$. ■

Let $\Lambda'(x,y)$ be the following first-order formula (cf. the usual definition of greatest fixed points as unions of "post-fixed points"):

$$\exists r(\mathrm{rel}(r) \wedge \langle x,y\rangle \in r \wedge r \subseteq \tau(r)).$$

Note that if $\langle x,y\rangle \in r \in M$, then (since $M \subseteq V_{n+1}$) $x,y \in V_{n-2}$. On the other hand, $\Lambda'$ *does* define over $M$ a linear ordering which includes the restriction of $\prec$ to $V_{n-3}$. This is clear from Claim 2.2 and by noting that, if $a,b \in V_{n-3}$ and $a \prec b$, then the relation $r = \{\langle x,y\rangle \mid x \prec y \wedge y \preceq b\}$ is in $M$ (for, it is in $V_n$) and witnesses that $M \models \Lambda'[a,b]$.

We extend the definition $\Lambda'$ to a definition $\Lambda$ on all of $M$, using the following pendant $T$ of $\tau$ that applies now to formulas instead of relations.

For an arbitrary formula $\varphi(x,y)$, let $T(\varphi)$ be the formula

$$\exists y' \in (y-x)\forall x' \in (x-y)\varphi(x',y').$$

Then, $T(\Lambda')$ defines over $M$ a linear ordering containing the restriction of $\prec$ to $V_{n-2}$, ... , and finally, putting $\Lambda = T(T(T(T(\Lambda'))))$, we have obtained our definition over $M$ of the restriction of $\prec$ to $M$. ∎

Theorem 2.1 has a corollary of interest from the perspective of descriptive complexity theory. Recall that the BIT relation on the set of natural numbers $N$ is defined as follows:

BIT$(m,n)$ iff there is a 1 in the $(m+1)^{\mathrm{st}}$ position in the binary representation of $n$.

Finite structures with the BIT relation as a "built-in relation" have been studied extensively in the context of descriptive complexity theory (see [Imm89]). It is well known that the structure $(N, \mathrm{BIT})$ is isomorphic to the structure $(V_\omega, \in)$ (see, for example, [Bar75]); indeed, the isomorphism carries the usual order relation on $N$ into the $\prec$ relation on $V_\omega$ defined above. The following result is now an immediate corollary of Theorem 2.1.

**Corollary 2.3.** $\mathrm{FO}(<, \mathrm{BIT}) = \mathrm{FO}(\mathrm{BIT})$.

## References


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Department of Computer Science, University of Wales, Swansea, Swansea SA2 8PP, U.K.

*E-mail address*: `a.dawar@swansea.ac.uk`

WINS, University of Amsterdam, Plantage Muidergracht 24, 1018TV Amsterdam, NL

*E-mail address*: `doets@wins.uva.nl`

Department of Mathematics, Haverford College, Haverford, PA 19041, U.S.A.

*E-mail address*: `slindell@haverford.edu`

Department of Philosophy, University of Pennsylvania, Philadelphia, Pennsylvania 19104, U.S.A.

*E-mail address*: `weinstein@cis.upenn.edu`