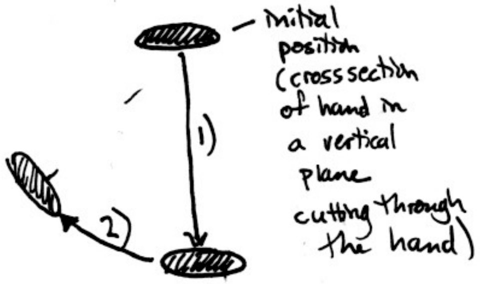
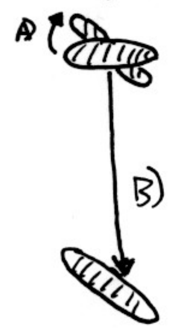
## Non-commutation of rotations

Stand facing into paper, arms extended straight in front of you, palms face down. Let x-axis come out of your shoulder & go straight ahead, while y-axis points left. Positive rotation ≡ ccl as viewed looking toward origin (your shoulder).

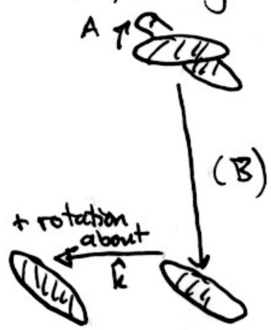
Left hand: 1) rotate +45° about $\hat{j}$
2) then +45° about $\hat{i}$

Right hand: A) rotate +45° about $\hat{i}$
B) rotate +45° about $\hat{j}$




The difference between these, i.e. $\hat{R}(\frac{\pi}{4}\hat{i})\hat{R}(\frac{\pi}{4}\hat{j}) - \hat{R}(\frac{\pi}{4}\hat{j})\hat{R}(\frac{\pi}{4}\hat{i})$ is mostly a rotation about $\hat{k}$: For example, rotating the right hand ccl about $\hat{k}$ gives




This is very similar to the final position of the left hand. In the limit of smaller rotations (rather than 45°), the similarity would be more exact, as you'll show on PS4.

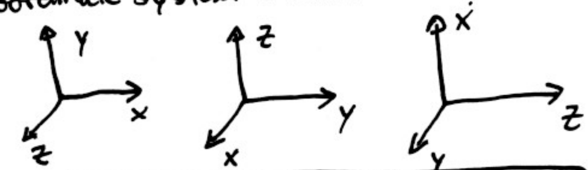
Because of the relationship between $\hat{J}_z$, $\hat{J}_x$, and $\hat{J}_y$ and rotations about the z, x, and y axes, it's reasonable based on the above to expect $[\hat{J}_x, \hat{J}_y] \propto \hat{J}_z$

You'll show on ps4 that, in fact

$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$. (Townsend shows this by a different method in section 3.1.)

## Permutations

Same coordinate system viewed from different angles:

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⇒ Can transform any result to an equivalent result by the cyclic permutation

x → y → z (→ x)

⇒
$$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$$
$$[\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x$$
$$[J_z, J_x] = i\hbar\hat{J}_y$$

(True for any system, not just electrons.)

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NB: An operator represented in the basis of its eigenstates is diagonal,

e.g. $\hat{J}_z \xrightarrow{z\text{-basis}} \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$\xrightarrow{x\text{-basis}} \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$

## The problem

We (& Townsend and the rest of the world) were a bit too cavalier with phase factors when determining $|+x\rangle$, $|+y\rangle$, etc.

For example:

z-basis
$$|+z\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad |+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad |+y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$$
$$|-z\rangle = \begin{pmatrix}0\\1\end{pmatrix} \quad |-x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \quad |-y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

x-basis
$$|+x\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |+y\rangle = \begin{pmatrix}\langle +x|+y\rangle\\ \langle -x|+y\rangle\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1+i\\1-i\end{pmatrix}$$
$$|-x\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\langle +x|+y\rangle: \ \frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{2}(1+i)$$
$$\langle -x|+y\rangle: \ \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{2}(1-i)$$

But, using cyclic permutation, should have
$$|+y\rangle \xrightarrow[\text{x-basis}]{} \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$$

Fixed version:

z-basis:
$$|+z\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad |+x\rangle = -\frac{e^{-i\pi/4}}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad |+y\rangle = -\frac{e^{i\pi/4}}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$$
$$|-z\rangle = \begin{pmatrix}0\\1\end{pmatrix} \quad |-x\rangle = \frac{e^{i\pi/4}}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \quad |-y\rangle = -\frac{e^{i\pi/4}}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

However, we'll continue to use the old (wrong versions). As long as we don't do cyclic permutations with them, they work fine.

## The $\hat{J}^2$ operator

$$\hat{J}^2 \equiv \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$$

$\rightarrow [\hat{J}_z, \hat{J}^2] = 0 \ \left(= [\hat{J}_y, \hat{J}^2] = [\hat{J}_x, \hat{J}^2]\right)$

$\Rightarrow$ can find a complete set of basis states that are simultaneous eigenfunctions of $\hat{J}_z$ & $\hat{J}^2$.

$$\hat{J}^2|\lambda, m\rangle = \lambda\hbar^2|\lambda, m\rangle$$
$$\hat{J}_z|\lambda, m\rangle = m\hbar|\lambda, m\rangle$$

where $\lambda$ & $m$ are dimensionless and are to be determined.

The best trick in physics: raising & lowering operators

$$\boxed{\hat{J}_\pm \equiv \hat{J}_x \pm i\hat{J}_y}$$

$$\rightarrow \hat{J}_+|\lambda, m\rangle = \begin{cases} c_+|\lambda, m+1\rangle \\ \text{-or-} \\ 0 \end{cases} \quad !!$$

$\Rightarrow$ There is a max. value of $m$ (call it $j$), as is required to avoid $J_z > J$

$$\hat{J}_+|\lambda, j\rangle = 0.$$

If $m<j$, then $\hat{J}_+|\lambda, m\rangle = c_+|\lambda, m+1\rangle$

$\Rightarrow$ The eigenstates of $\hat{J}_z$ have values of $s_z$ spaced by $\hbar$ !!

## Example: Spin 1 particles

If we do the SG expt on some particles (not electrons), we find $S_z = -\hbar, 0,$ or $\hbar$

$$S_z = \hbar \longleftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad S_z = 0 \longleftrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad S_z = -\hbar \longleftrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \hat{J}_z \xrightarrow[\text{z-basis}]{} \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

You'll show on ps4 that

$$\hat{J}_x \xrightarrow[\text{z-basis}]{} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \hat{J}_y \xrightarrow[\text{z-basis}]{} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$